The Boston Area Undergraduate Physics Competition

April 22, 1995, 1:00-4:00 pm
Jefferson 250
Harvard University

Each of the six questions is worth 10 points.

1. A cylindrical pipe is positioned with its axis parallel to the ground. The radius is r and the axis is at height h above the ground. What is the minimum initial speed a ball must have in order to be thrown over the pipe? (The ball is thrown from the ground, $h = 0$, and has negligible size.) Consider two cases:

   (a) The ball is allowed to touch the pipe (you may assume the pipe is frictionless).

   (b) The ball is not allowed to touch the pipe.

2. These three sub-questions require only short, one or two sentence, answers.

   (a) A container is divided into two equal parts by a partition. One part contains an ideal gas at temperature T, the other part is a vacuum. The partition is quickly removed. What is the new equilibrium temperature of the gas in the container?

   (b) A semi-infinite wire, carrying a current I, ends on a infinite conducting plane that is perpendicular to the wire. Find the magnetic field at a distance r from the wire and a distance d from the plane.

   (c) A hockey puck, sliding on a frictionless surface, is attached by a piece of string (lying on the surface) to a vertical pole. The puck is given a tangential velocity, and as the string wraps around the pole, the puck spirals in. Explain what, if anything, is wrong with the following statement: "From conservation of angular momentum, the speed of the puck will become greater as the distance from the pole decreases. Hence the kinetic energy increases, and energy is not conserved."

3. Only a single wire is necessary to make a telegraph connection. Both terminals may have metalic objects buried in the ground, making the earth the other wire. Assume that these objects are spheres of radius r buried very deep in the ground (ignore edge effects due to the finite size of the depth). The distance between the terminals is L. What is the resistance (due to the earth) between the terminals? Assume that the resistivity ρ of the earth is uniform, and that $L \gg r$.

4. The semi-infinite circuit shown below is connected to an oscillating emf of the form $V_0 \cos \omega t$. Each inductor has inductance L, and each capacitor has capacitance C.

(a) What is the current (as a function of time) through the leads A, B?

(b) What is the average power delivered by the emf source?

5. A mountain climber wishes to climb a frictionless conical mountain. He wants to do this by throwing a lasso (a rope with a loop) over the top and climbing up along the rope (assume the mountain climber is of negligible height, so that the rope lies along the mountain). At the bottom of the mountain are two stores, one which sells "cheap" lassos (made of a segment of rope tied to loop of rope of *fixed* length), and the other which sells "deluxe" lassos (made of one piece of rope with a loop of *variable* length; the loop's length may change without any friction of the rope with itself).

   When viewed from the side, this conical mountain has an angle α at its peak. For what angles α can the climber climb up along the mountain if he uses:

   (a) a "cheap" lasso and loops it once around the top of the mountain?

   (b) a "deluxe" lasso and loops it once around the top of the mountain?

   (c) a "cheap" lasso and loops it N times around the top of the mountain? (Assume no friction of the rope with itself.)

   (d) a "deluxe" lasso and loops it N times around the top of the mountain? (Assume no friction of the rope with itself.)

6. This problem deals with the terminal velocity of a pencil rolling down an inclined plane. To avoid cumbersome calculations of moments of inertia, in this problem we will approximate the pencil as having all its mass M located on the center axis. To avoid other complications, we will assume that the cross section of the pencil looks like a wheel with six equally spaced spokes and no rim. The lengths of these spokes are all r. (An uncomfortable pencil to handle, indeed; but a much more comfortable one to theorize about.)

   The angle of inclination of the plane is α. Assume there is infinite friction (no slipping) between the pencil and the plane, and that the pencil does not bounce when the end of a "spoke" hits the plane. Assume the plane is very hard so that the "spokes" press into the plane a negligible amount.

   (a) Explain why the speed of the pencil is bounded from above (assuming that it remains in contact with the plane at all times), i.e., why it reaches some (average) terminal velocity.

   (b) Assume that conditions are set up such that the pencil will eventually reach some non-zero terminal (average) velocity (while remaining in contact with the plane at all times). Describe this terminal velocity; you may do this by simply stating the maximum speed of the axis of the pencil in this "steady" state.

(c) What is the minimum angle of inclination α which allows a non-zero terminal velocity to exist? (An initial kick to the pencil is allowed.)

(d) What is the maximum angle of inclination α if the pencil is to stay in contact with the plane at all times?

(e) Do parts (b), (c), and (d) for a pencil with N equally spaced "spokes", where N is very large. In addition to assuming N is very large, you may assume the angle of inclination α is small, and you may use small angle approximations ($\sin\theta \approx \theta$, etc.) where appropriate.

(f) For very large N, what is the maximum possible terminal velocity if the pencil is to remain in contact with the plane at all times?

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SOLUTIONS

1. (a) Imagine the ball sitting on top of the pipe. If it is given a tiny push, it will slide off the pipe and hit the ground with a speed given by $\frac{1}{2}mv^2 = mg(r+h)$. This motion may be reversed. The ball must therefore be thrown with a speed of just greater than $\sqrt{2g(r+h)}$. By conservation of energy, clearly no smaller speed will work.

   (b) *First Solution:* (Based on a solution by Charles Santori and Ron Maimon) Let the parabolic arc of the ball be tangent to the pipe at an angle θ from the top of the pipe. The velocity of the ball there is of the form $(v_\theta \cos\theta, v_\theta \sin\theta)$. The conditions that the parabola reach its maximum over the center of the pipe (any other situation would require more energy) are

$$(v_\theta \cos\theta)t = r\sin\theta, \qquad \text{and} \qquad v_\theta \sin\theta = gt,$$

for some t. These give

$$v_\theta^2 = \frac{gr}{\cos\theta}.$$

Let v be the speed at which the ball is thrown from the ground. Then the energy at the ground is

$$\frac{1}{2}mv^2 = \frac{1}{2}m\Big(\frac{gr}{\cos\theta}\Big) + mg(h + r\cos\theta).$$

Minimizing this function of $\cos\theta$ yields $\cos\theta = 1/\sqrt{2}$, and

$$v^2 = g(2\sqrt{2}r + 2h).$$

Note that $\theta = 45^\text{o}$, independent of the ratio of h to r .

Second Solution: Let the parabolic arc of the ball have a maximum height l and span a distance $2d$ on the ground. If the initial velocity of the ball is (v_x, v_y) and it is in the air for time $2t$, then

$$v_x t = d, \qquad v_y = gt, \qquad \text{and} \qquad \frac{1}{2}gt^2 = l.$$

Eliminating t gives

$$v^2 = g(\frac{d^2}{2l} + 2l).$$

We want to minimize this, given that the parabola goes over the pipe. The parabola is given by

$$y = -\frac{l}{d^2}x^2 + l.$$

The pipe's cross section is given by

$$(y-h)^2 + x^2 = r^2.$$

Solving for the y value of their intersection gives

$$y = \frac{1}{2l}(2lh + d^2 \pm \sqrt{d^4 + d^2(4lh - 4l^2) + 4l^2r^2}).$$

We want the parabola to be tangent to the pipe, i.e., the discriminant here is zero. Therefore, $d^2 = 2l(l - h \pm \sqrt{(l-h)^2 - r^2})$. The minus sign is the physically relevant one (the plus sign corresponds to negative values for x^2). The above expression for v^2 now yields

$$v^2 = g(3l - h - \sqrt{(l-h)^2 - r^2}).$$

Minimizing this function of l yields $l = h + \frac{3}{2\sqrt{2}}r$, and

$$v^2 = g(2\sqrt{2}r + 2h).$$

[Note that this is indeed less than the value at $l = h + r$, namely $v^2 = g(3r + 2h)$, a common, incorrect answer to this problem. Touching the pipe tangentially at the top is a limiting case of having two tangent points very close together. A possible candidate for an optimal path not covered in the above solution is one which is tangent to the top of the pipe, and which has a larger value for d than the one just mentioned. But we know that v^2 increases with d, for constant l.]

2. (a) If the partition is removed very quickly, no work is done on the gas. Therefore, the temperature remains equal to T.

   (b) Due to the cylindrical symmetry, the component of B in the tangential direction around the wire is, from Ampere's Law, $\frac{\mu_o I}{2\pi r}$. ¿From a few applications of the right hand rule, it is easy to see that the currents in the plane will give only tangential components.

   (c) Energy is conserved, but the angular momentum of the puck is not. The force from the string is not a central force; or rather, the "center" keeps changing. (Angular momentum *is* conserved, of course, if the earth is included.)

3. It is given that $L \gg r$. Consider a sphere of radius aL $(L \gg aL \gg r)$ surrounding one of the metal spheres. Since $L \gg aL$, the current flow is approximately spherically symmetric out to aL. The resistance out to aL is computed by considering many spherical shells of thickness dr in series. The resistance of a shell is $\rho dr/(4\pi r^2)$. Integrating this gives a resistance of $\frac{\rho}{4\pi}(\frac{1}{r} - \frac{1}{aL})$ out to a radius aL. The other sphere has the same resistance out to a radius aL. So the total resistance between the two metal spheres is $\frac{\rho}{2\pi}(\frac{1}{r} - \frac{1}{aL})$ plus the resistance between the spheres of radius aL. This latter resistance is less than that of a cylinder of radius aL and length L, which has a resistance of $\rho L/(\pi a^2 L^2)$. This is negligible compared to the $1/r$ term

above, as long as $a^2L \gg r$. If L is large enough compared to r, then it is possible to pick an a so that $L \gg aL \gg a^2L \gg r$. Therefore, keeping only the leading term,

$$R = \frac{\rho}{2\pi r}.$$

4. (a) We will use complex impedances to solve this problem, and use the equation analogous to Ohm's law: $I = \mathrm{Re}(V_o e^{i\omega t}/Z)$, where Z is the complex impedance. Since we have a semi-infinite circuit here, adding another "box" to the circuit shouldn't change the impedance. Therefore,

$$Z = i\omega L + \frac{Z\frac{1}{i\omega C}}{Z + \frac{1}{i\omega C}}.$$

Solving for Z gives

$$Z = \frac{1}{2}\left(i\omega L \pm L\sqrt{\frac{4}{LC} - \omega^2}\right).$$

There are two cases to consider:

i. $\omega > 2/\sqrt{LC}$:
The impedance is purely imaginary,

$$Z = \frac{i}{2}\left(\omega L + L\sqrt{\omega^2 - \frac{4}{LC}}\right).$$

The plus sign is selected because for $C \to \infty$, Z must become $i\omega L$ (alternatively, for $\omega \to \infty$, Z must go to ∞). $I = \mathrm{Re}(V_o e^{i\omega t}/Z)$ now gives

$$I(t) = \frac{2V_o}{\omega L + L\sqrt{\omega^2 - \frac{4}{LC}}}\sin\omega t.$$

ii. $\omega < 2/\sqrt{LC}$:
The impedance is

$$Z = \frac{1}{2}\left(i\omega L + L\sqrt{\frac{4}{LC} - \omega^2}\right).$$

The plus sign is chosen, since the real part of an impedance is positive. $I = \mathrm{Re}(V_o e^{i\omega t}/Z)$ now gives

$$I(t) = \frac{V_o}{\sqrt{L/C}}\cos(\omega t - \phi),$$

where $\tan\phi = \omega/\sqrt{\frac{4}{LC} - \omega^2}$.

(b) i. $\omega > 2/\sqrt{LC}$:

¿From $V(t) = V_o \cos\omega t$ and the expression for I above, we see that the average power delivered by the source goes like the average of $\sin\omega t \cos\omega t$, so

$$\langle P \rangle = 0$$

(the real part of the impedance is zero).

ii. $\omega < 2/\sqrt{LC}$:

The average of $P = IV$ is the average of $V_o \cos\omega t$ times the expression for I given above, so we obtain

$$\langle P \rangle = \frac{1}{4} C V_o^2 \sqrt{\frac{4}{LC} - \omega^2}.$$

This is non-zero because the impedance has a real part. The power delivered by the source would be zero for a *finite* circuit. In our infinite case, the energy of the source is dissipated in the form of a wave propagating along the circuit.

5. (a) This will be a little wordy since we're going to try to do this without drawing any pictures. Cut the cone along a straight line emanating from the peak and passing through the knot of the lasso, and roll the cone flat onto a plane. Call the resulting figure, a sector of a circle, S. (You may want to stop reading at this point, and try to solve it yourself.)

If the cone is very sharp, then S will look like a thin "pie piece". If the cone is very wide, with a shallow slope, then S will look like a pie with a piece taken out of it. Points on the straight-line boundaries of S are identified. Let P be the location of the lasso's knot. Then P appears on each straight-line boundary, at equal distances from the tip of S. Let β be the angle of the sector S.

The path of the lasso's loop must be a straight line on S. (The rope will take the shortest distance between two points, since there is no friction, and rolling the cone onto a plane does not change distances.) Such a straight line between the two identified points P is possible only if the sector S is smaller than a semicircle, i.e., $\beta < 180^o$.

Let C denote a cross sectional circle a distance d from the top of the conical mountain, and let R equal the ratio of the circumference of C to d. Then a semicircle S implies that $R = \pi$. This then implies that the radius of C is equal to $d/2$. Therefore, $\alpha/2 = \sin^{-1}(1/2)$. So we find that if the climber is to be able to climb up along the mountain, then

$$\alpha < 60^o.$$

Having $\alpha < 60^o$ guarantees that there is a loop around the cone of shorter length than the distance straight to the peak and back.

[When viewed from the side, the rope should appear perpendicular to the side of the mountain at the point opposite the lasso's knot. A common mistake is to assume that this implies $\alpha < 90^o$. This is not the case because the loop does not lie in a plane. Lying in a plane, after all, would imply an elliptical loop; but the loop must certainly have a discontinuous change in slope where the knot is. (For planar, triangular mountains, the answer is $\alpha < 90^o$.)]

(b) Same strategy. Roll the cone onto a plane. If the mountain very steep, the climber's position can fall by means of the loop growing larger; if the mountain has a shallow slope, the climber's position can fall by means of the loop growing smaller. The only situation in which the climber will not fall is the one where the change in the position of the knot along the mountain is exactly compensated by the change in length of the loop.

In terms of the figure S on a plane, the condition is that if we move P a distance l up (down) along the mountain, the distance between the identified points P decreases (increases) by l. We must therefore have $2\sin(\beta/2) = 1$. So $\beta = 60^o$, which corresponds to

$$\alpha = 2\sin^{-1}(1/6).$$

There is exactly one angle for which the climber can climb up along the mountain.

Another way to see that $\beta = 60^o$ is to note that the three directions of rope emanating from the knot must all have the same tension, since the deluxe lasso is one continuous piece of rope. Therefore they must have 120^o angles between themselves. This implies that $\beta = 60^o$.

(c) Roll the cone N times onto a plane. The resulting figure S_N is a sector of a circle divided into N equal sectors, each representing a copy of the cone. S_N must be smaller than a semicircle, so we must have $R < \pi/N$. Therefore,

$$\alpha < 2\sin^{-1}(\frac{1}{2N}).$$

(d) Roll the cone N times onto a plane. From the above reasoning, we want $N\beta = 60^o$. Therefore,

$$\alpha = 2\sin^{-1}(\frac{1}{6N}).$$

6. (a) The main point of this problem is that when the pivot point of the pencil changes (i.e., when a new spoke hits the plane), the speed of the axis changes suddenly, and kinetic energy is lost. Only the velocity component perpendicular to the new spoke survives from the previous velocity (which was perpendicular to the old spoke). The loss in kinetic energy is proportional to the square of the velocity right before the change. When the speed has increased to a magnitude where this loss in kinetic energy equals the gain from the change in potential energy, the pencil will not go any faster.

(b) We may as well do the problem for a general number of spokes, N, and then let $N = 6$. Let α be the angle of inclination of the plane, v_o be the speed of the axis right before a new spoke hits, and $\beta = 2\pi/N$. Then the speed of the axis right after the new spoke hits is $v_o \cos\beta$.

Equating the change in potential energy during an Nth of a rotation and the kinetic energy loss due to the changing of the contact spoke gives $\frac{1}{2}mv_o^2(1-\cos^2\beta) = mgr(2\sin\frac{\beta}{2})\sin\alpha$. So in the "steady" state, the maximum speed v_o of the axis is given by

$$v_o^2 = \frac{4gr\sin\frac{\beta}{2}\sin\alpha}{\sin^2\beta}.$$

For $N = 6$ and $\beta = \pi/3$, we have

$$v_o^2 = \frac{8}{3}gr\sin\alpha.$$

If conditions have been set up (assuming contact is maintained with the plane, as stated in the problem) such that a non-zero v_o exists, it must be this.

(c) If $\alpha < \beta/2$, then right after the pivot point changes, the axis must actually move upward before falling down along the plane. For a non-zero v_o to exist, we must ensure that the axis is moving fast enough to get over this "bump" (remember that an initial kick to the pencil is allowed). So (assuming $\alpha < \beta/2$) the height the axis must climb is $r(1-\cos(\frac{\beta}{2}-\alpha))$. The speed at which the axis starts this climb is $v_o\cos\beta$. Therefore, we must have $\frac{1}{2}m(v_o\cos\beta)^2 > mgr(1-\cos(\frac{\beta}{2}-\alpha))$. Using the expression for v_o above,

$$\frac{2\sin\frac{\beta}{2}\sin\alpha\cos^2\beta}{\sin^2\beta} > (1-\cos(\frac{\beta}{2}-\alpha)).$$

For $N = 6$ and $\beta = \pi/3$, we have

$$\frac{\sqrt{3}}{2}\cos\alpha > 1 - \frac{5}{6}\sin\alpha.$$

Squaring and solving for $\sin\alpha$ gives

$$\sin\alpha > \frac{15-6\sqrt{3}}{26}.$$

(One may estimate this using $\sin\alpha \approx \alpha$ to obtain an angle of about 10^o.)

(d) The axis of the pencil moves on a circular arc around the pivot point. The force of gravity along the contact spoke must account for the centripetal acceleration of the axis.

The maximal centripetal acceleration occurs right before the pivot point changes, and is equal to mv_o^2/r. The minimal force along the spoke from gravity also occurs right before the pivot point changes, and is $mg\cos(\alpha + \frac{\beta}{2})$. Using the expression for v_o above, $mv_o^2/r \leq mg\cos(\alpha + \frac{\beta}{2})$ becomes

$$\tan\alpha \leq \frac{\sin^2\beta}{4 + \sin^2\beta}\cot(\beta/2).$$

For $N = 6$ and $\beta = \pi/3$, this gives

$$\tan\alpha \leq \frac{3\sqrt{3}}{19}.$$

(A small angle approximation shows this to be about 15°.)

(e) For small β and α we find, using the expressions in (b), (c), and (d):

The expression for v_o becomes

$$v_o^2 = 2gr\frac{\alpha}{\beta}.$$

The condition to make it over the "bump" becomes

$$\frac{\alpha}{\beta} > \frac{1}{2}(\alpha - \frac{\beta}{2})^2.$$

For small α and β this implies $\alpha > 0$ (up to third order corrections).

The condition to stay on the plane becomes

$$\alpha \leq \beta/2.$$

In other words, if the angle of inclination is increased until the pencil starts to roll on its own, then it will eventually leave the plane.

(f) Combining the large-N answers to (b) and (d) gives

$$v_o \leq \sqrt{gr},$$

which is independent of N.

This last result can be obtained in a simpler way, which makes it less surprising. An inverted pendulum's centripetal acceleration mv^2/r must be accounted for by the weight mg on the spoke. Therefore $v^2 \leq gr$. The tilt of the plane will change this by factors essentially equal to 1, for small α.

The Boston Area Undergraduate Physics Competition

April 20, 1996, 1:00-4:00 pm
Jefferson 250
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Each of the six questions is worth 10 points.

1. (a) A ball, B_2, with (very small) mass m_2 sits on top of another ball, B_1, with (very large) mass m_1 [see the figure below]. The bottom of B_1 is at a height h above the ground, and the bottom of B_2 is at a height $h + d$ above the ground. The balls are dropped. How high does the top ball bounce?

   (You may work in the approximation where m_1 is much heavier than m_2. Assume that the balls bounce elastically. And assume, for the sake of having a nice clean problem, that the balls are initially separated by a small distance, and that the balls bounce instantaneously.)

   (b) n balls, $B_1, \ldots, B_n$, with masses $m_1, m_2, \ldots, m_n$, respectively ($m_1 \gg m_2 \gg \cdots \gg m_n$), sit in a vertical stack [see the figure below]. The bottom of B_1 is at a height h above the ground, and the bottom of B_n is at a height $h + \ell$ above the ground. The balls are dropped. In terms of n, how high does the top ball bounce?

   (Work in the approximation where m_1 is much heavier than m_2, which is much heavier than m_3, etc., and assume that the balls bounce elastically. Also, make the "nice clean problem" assumptions as in part (a).)

   If $h = 1$ meter, what is the minimum number of balls needed in order for the top one to bounce to a height of at least 1 kilometer?

   (Assume the balls still bounce elastically [which is in reality not likely]. Ignore wind resistance, etc., and assume that l is negligible here.)

2. Each of the two unrelated parts of this question is worth 5 points:

   (a) Let $\vec{E}$ be the electric field due to a circular ring of radius R and uniform charge density. Show that somewhere along the surface of a cylinder of radius $R/2$, whose axis passes through the center of the ring and is perpendicular to the plane of the ring [see the figure below], $\vec{E}$ is parallel to the axis of the cylinder.

   (b) Two cylindrical containers, A and B, have the same shape and contain equal volumes of water. In addition to the water, B contains an immersed balloon, attached to the bottom with a string [see the figure below]. (Assume that the mass of the air in the balloon is negligible.) The following reasoning claims that the pressure in the water at the bottom of A is the equal to the pressure in the water at the bottom of B. Is this reasoning correct or incorrect? (Explain why)

   Reasoning: The total upward force exerted by the bottom of container A is equal to the weight of the water in A; likewise for B. But the force exerted by the bottom of A is equal to the pressure times the area of the bottom; likewise for B. Therefore, since the areas of the bottoms are the same, and the weights of the water are the same, the pressures at the bottoms must be the same.

3. In the figure below, the vertices $ABCDEFGH$ form a cube. Each of the twelve edges and each of the twelve diagonals on the surface (two diagonals on each of the six faces) is a 1Ω resistor. (In other words, if ℓ is the length of an edge, then every segment of length ℓ or $\sqrt{2}\,\ell$ is a 1Ω resistor. We have drawn a few of the diagonal resistors in the figure, but we have not drawn them all, for the sake of having a readable figure.)

   Find the resistance between points A and C.

4. (a) Two circular rings, in contact with each other, stand in a vertical plane [see the figure below]. Each has radius R. A small ball, with mass m and negligible size, bounces elastically back and forth between the rings. (Assume that the rings are held in place, so that they always remain in contact with each other.) Assume that initial conditions have been set up so that the ball's motion forever lies in one parabola. Let this parabola hit the rings at an angle θ from the horizontal.

   i. Let $\Delta P_x(\theta)$ be the magnitude of the change in the horizontal component of the ball's momentum, at each bounce. For what angle θ is $\Delta P_x(\theta)$ maximum?

   ii. Let S be the speed of the ball just before or after a bounce. And let $\overline{F}_x(\theta)$ be the average (over a long period of time) of the magnitude of the horizontal force needed to keep the rings in contact with each other (for example, the average tension in a rope holding the rings together). Consider the two limits: (1) $\theta \approx \epsilon$, and (2) $\theta \approx \pi/2 - \epsilon$, where ϵ is very small.

      A. Derive approximate formulas for S, in these two limits.

      B. Derive approximate formulas for $\overline{F}_x(\theta)$, in these two limits.

      (You may use the small-angle approximations $\sin\epsilon \approx \epsilon$, $\cos\epsilon \approx 1 - \epsilon^2/2$, etc. Give your answers to leading order in ϵ.)

(b) Consider the more general case where a ball bounces back and forth between a surface defined by $f(x)$ (for $x > 0$) and $f(-x)$ (for $x < 0$) [see the figure below]. Again, assume that initial conditions have been set up so that the ball's motion forever lies in one parabola (the ball bounces back and forth between the contact points at $(x_0, f(x_0))$ and $(-x_0, f(x_0))$, for some x_0).

   i. Let $\Delta P_x(x_0)$ be the absolute value of the change in the horizontal component of the ball's momentum, at each bounce. For what function $f(x)$ is $\Delta P_x(x_0)$ independent of the contact position x_0?

   ii. Let $\overline{F}_x(x_0)$ be the average of the magnitude of the horizontal force needed to keep the two halves of the surface together. For what function $f(x)$ is $\overline{F}_x(x_0)$ independent of the contact position x_0?

5. Consider the following rigid structure: A mass M is located at the vertex of an angle formed by two sticks [see the figure below], each having length ℓ and negligible mass. The fixed angle between the sticks is θ.

   In this problem you are asked to determine, for small θ, approximately how long this structure can rock back and forth.

   More concretely: hold the structure so that one of the sticks (say, the left one) is vertical. Then give the mass an infinitesimal push, so that eventually the right stick will hit the ground. Let this occur at time t_0. (Assume that there is sufficient friction between the sticks and the ground so that the sticks do not slide, and assume that the sticks do not bounce when they hit the ground.) The left stick will lose contact with the ground, and the system will then pivot around the right stick; the mass will first rise, and then fall until the left stick hits the ground, etc. Eventually this rocking motion will cease, and the system will come to a halt. Let this occur at time T.

   For small θ, calculate, to leading order in $1/\theta$, the value of $T - t_0$.

   (You may use the small-angle formulas, $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$.)

   [Helpful hints in simplifying your answer:

   (a) $\ln(1-x) = -(x + x^2/2 + x^3/3 + x^4/4 + \cdots)$ for $-1 < x < 1$,

   (b) $1 + 1/3^2 + 1/5^2 + 1/7^2 + \cdots = \pi^2/8$.

   You get 7 points for getting a correct expression for $T - t_0$, and 3 points for simplifying it to show the explicit leading behavior in $1/\theta$; so you may not want to spend too much time on the simplification.]

6. A block with mass M (M is very large) initially sits on a frictionless plane, at a distance of 1 meter from the top of the plane [see the figure below]. The angle of inclination of the plane is θ. An elastic band connects the base of the block to the top of the plane. A cylindrical object, C, with mass m (m is very small), sits on the elastic band where it is connected to the plane. (The axis of C is parallel to the ground.)

   At $t = 0$, the block is allowed to slide down the plane, and C is allowed to roll down the elastic band.

   Assume: (1) the elastic band stretches uniformly, (2) M is large enough, and the elastic band is weak enough, so that the band has no effect on the motion of the block; the only purpose of the elastic band in this problem is to provide a surface for C to roll on (and the band, since it is stretching, may drag C along a bit), (3) there is sufficient friction between C and the elastic band so that C rolls on the band without slipping, and (4) m is small enough so that C has no effect on the stretching of the band.

   Let C have radius r and moment of inertia $I = \rho m r^2$.

   (a) (2 points) Show that if $\rho = 0$ (i.e., $I = 0$), then C will always remain 1 meter away from the block.

   (b) (8 points) Show that if $\rho > 0$ (i.e., $I > 0$), then

      i. the ratio of C's distance travelled along the plane to the block's distance travelled along the plane approaches 1, as $t \to \infty$;

      ii. as a function of t, the relative speed of the block and C behaves like $G(\rho)t^{-(1-\rho)/(1+\rho)}$, as $t \to \infty$, where $G(\rho)$ is a function of ρ which you are *not* required to determine.

1996 Boston Area Undergraduate Physics Competition

SOLUTIONS

1. (a) Just before B_1 hits the ground, both balls are moving downward with a velocity obtained via $mv^2/2 = mgh$; thus $v = \sqrt{2gh}$. Just after B_1 hits the ground, it moves upward with a speed v, while B_2 is still moving downward with speed v. The relative speed is therefore $2v$. Hence, after the balls bounce off each other, the relative speed is still $2v$ (as can be seen by working in the frame of reference where B_1 is motionless, and using $m_1 \gg m_2$). Since the speed of B_1 essentially stays equal to v, the upward speed of B_2 is therefore $2v + v = 3v$. By conservation of energy, it will therefore rise to a height of $H = d + (3v)^2/(2g)$, or

$$H = d + 9h. \tag{1}$$

(b) Just before B_1 hits the ground, all of the balls are moving downward with a velocity obtained via $mv^2/2 = mgh$; thus $v = \sqrt{2gh}$.

Let us inductively determine the speed of each ball after it bounces off the one below it. If B_i achieves a speed of v_i after bouncing off B_{i-1}, then what is the speed of B_{i+1} after it bounces off B_i? Well, the relative speed of B_{i+1} and B_i (right before they bounce) is $v + v_i$. This is also the relative speed after they bounce. The final upward speed of B_{i+1} is therefore $(v + v_i) + v_i$, so

$$v_{i+1} = 2v_i + v. \tag{2}$$

Since $v_1 = v$, we obtain $v_2 = 3v$ (in agreement with part (a)), $v_3 = 7v$, $v_4 = 15v$, etc. In general,

$$v_n = (2^n - 1)v, \tag{3}$$

which is easily seen to satisfy eq. (2), with the initial value $v_1 = v$.

¿From conservation of energy, B_n will bounce to a height of $H = l + (2^n - 1)^2 v^2/(2g)$, or

$$H = l + (2^n - 1)^2 h. \tag{4}$$

If h is 1 meter, and we want this height to equal 1000 meters, then (assuming l is not very large) we need $2^n - 1 > \sqrt{1000}$. Five balls won't quite do the trick, but six will, and in this case the height is almost four kilometers.

[Escape velocity from the earth is reached when $n = 14$. Of course, the elasticity assumption is absurd in this case, as is the notion that one may find 14 balls with the property that $m_1 \gg m_2 \gg \cdots \gg m_{14}$.]

2. (a) It is fairly easy to see that at a point P far away from the ring the electric field must point slightly away from the axis (assuming that the ring is positively charged). This is because the distances from different points on the ring to P differ only by small second-order effects (on the order of the ratio of R^2 to the square of the distance from the ring to P), whereas the radial components of the different contributions to the field at P differ by first-order effects of this ratio.

Now, if we can demonstrate the fact that the field points inward toward the axis at some point on the cylinder, then we can simply invoke continuity of the field (since it points outward far away) to say that at some point the field lies along the cylinder. This demonstration can be done with a messy calculation for points in the plane of the ring, but it's easier to just use Gauss' Law: The total flux through the cylinder must be zero; therefore, given that it points outward somewhere (and that it obviously points outward at the ends of an imaginary cylindrical Gaussian surface), it must point inward somewhere.

(b) The reasoning is incorrect; the total force exerted by the bottom of B is *not* equal to the pressure times the area of the bottom. The bottom is exerting a force on the string (downward). Thus the pressure is actually a bit higher, since the sum of all the forces exerted by the bottom must indeed be the weight of the water.

[This is consistent with the fact that a little piece of area on the bottom of B has to support a taller column of water than one on the bottom of A. But just giving this column-of-water reasoning only says that the given reasoning is incorrect; it doesn't explain why it is incorrect.]

3. *First Solution:* The key is to realize that if a voltage difference is applied between points A and C, then points B, D, F, and H are all at the same potential (each one is, by symmetry, at a potential equal to the average of the potentials at A and C). Therefore, we can bring these four points together, without changing the resistance between A and C. We may then put the whole network in a plane, with A, C, G, and E as the vertices of a square, and one point (representing B, D, F, and H) in the middle. Each side of this square is a 1Ω resistor, and there are also four 1Ω resistors going from each of A, C, G, and E to the point in the middle. (The four remaining resistors out of the original twenty-four connect the identified points B, D, F, and H.)

We therefore equivalently have a square of 1Ω resistors, and a resistor of $\frac{1}{4}\Omega$ going from each vertex to a point in the middle. We may then identify the midpoint of AC with the point in the middle (since they are at the same potential); likewise for the midpoint of EG. This process essentially adds on a resistor of $\frac{1}{2}\Omega$ from each of A, C, G, and E to the center, while eliminating sides AC and EG of the square. So now we effectively have a 1Ω resistor between A and E, a 1Ω resistor between C and G, and $\frac{1}{6}\Omega$ resistors from each of A, C, G, and E to the middle. It is easy to see that this yields a resistance of $\frac{7}{24}\Omega$ between A and C.

Second Solution (due to Kiran Kedlaya): Let A be at a voltage of $+1$, and C be at a voltage of -1 (we may pick these to have magnitude 1, without loss of generality). Then by symmetry, the voltages of B, D, F, and H are all 0. Also, the voltages of E and G are $+x$ and $-x$, respectively, where x is to be determined.

Now, the current along any of the resistors is simply equal to the voltage difference of the endpoints (using $I = V/R$, with $R = 1\Omega$). So the sum of the currents going into, say, E (which come from the six points A, F, H, B, D, and G) is

$$I_{\to E} = (1-x) + (0-x) + (0-x) + (0-x) + (0-x) + (-x-x) = 1 - 7x. \qquad (5)$$

But this sum must be 0. Hence $x = \frac{1}{7}$.

It is then easy to see, by the same reasoning, that the current flowing out of A (or into C) is $7 - x = \frac{48}{7}$.

Since the voltage difference between A and C is 2, and the current between them is $\frac{48}{7}$, the resistance between them must be $\frac{7}{24}\Omega$.

4. (a) i. Let us find the velocity (v_x, v_y) of the ball just after it bounces off the left ring. If the ball takes a time t to hit the other ring, then we must have

$$2v_y = gt. \qquad (6)$$

Also, if the bounces on the rings occur at an angle of θ from the horizontal, then the horizontal distance between contact points is $2R(1-\cos\theta)$. Hence,

$$v_x t = 2R(1-\cos\theta). \qquad (7)$$

Eliminating t from the previous two equations, and using $v_y = v_x \tan\theta$, gives

$$(v_x, v_y) = \sqrt{gR}\left(\sqrt{\cot\theta(1-\cos\theta)}, \sqrt{\tan\theta(1-\cos\theta)}\right). \qquad (8)$$

Now, $\Delta P_x = 2mv_x$, so to maximize ΔP_x, we want to maximize $\cot\theta(1-\cos\theta)$. Setting the derivative of this equal to zero yields

$$\cos^3\theta - 2\cos\theta + 1 = 0. \qquad (9)$$

An obvious root is $\cos\theta = 1$. The two other roots are then $\cos\theta = (-1 \pm \sqrt{5})/2$. The root of 1 is not the one we want, since $\Delta P_x \propto v_x \to 0$ as $\theta \to 0$ (since $1-\cos\theta \approx \theta^2/2$, and $\cot\theta \approx 1/\theta$). So the maximum must occur at

$$\cos\theta = \frac{-1+\sqrt{5}}{2}. \qquad (10)$$

(This happens to be the inverse of the golden ratio. θ is about 52°.)

ii. A. Using eq. (8), along with $\tan\epsilon \approx \epsilon$ and $\cos\epsilon \approx 1 - \epsilon^2/2$ for small ϵ, we have:

- If $\theta = \epsilon$, with $\epsilon \to 0$, then

$$(v_x, v_y) = \sqrt{gR/2}\,(\epsilon^{1/2}, \epsilon^{3/2}). \tag{11}$$

To leading order in ϵ, the speed therefore goes like $S = \sqrt{gR\epsilon/2}$, which goes to zero, as $\epsilon \to 0$.

- If $\theta = \pi/2 - \epsilon$, with $\epsilon \to 0$, then

$$(v_x, v_y) = \sqrt{gR}\,(\epsilon^{1/2}, \epsilon^{-1/2}). \tag{12}$$

To leading order in ϵ, the speed therefore goes like $S = \sqrt{gR/\epsilon}$, which goes to infinity, as $\epsilon \to 0$.

B. Force is the change in momentum per time, so the average horizontal force, $\overline{F}_x$, needed to keep the rings together, is $\overline{F}_x = \Delta P_x/T$ where T the time between bounces. Since $\Delta P_x = 2mv_x$ and $T = 2v_y/g$, we see that $\overline{F}_x = \Delta P_x/T = mgv_x/v_y$. Hence,

$$\overline{F}_x = mg\cot\theta. \tag{13}$$

Therefore

- If $\theta = \epsilon$, with $\epsilon \to 0$, then

$$\overline{F}_x = mg\cot\epsilon \approx mg/\epsilon. \tag{14}$$

- If $\theta = \pi/2 - \epsilon$, with $\epsilon \to 0$, then

$$\overline{F}_x = mg\cot(\pi/2 - \epsilon) \approx mg\epsilon. \tag{15}$$

In the first limit, even though the speed of the ball approaches zero, the average horizontal force becomes infinite as $\theta \to 0$, because the bounces are so frequent. In the second limit, the speed of the ball approaches infinity, but the average horizontal force goes to zero as $\theta \to \pi/2$.

[Using the same reasoning, we see that the average vertical force is $F = \Delta P_x/T = mgv_y/v_y = mg$. This does not depend on the angle, which is the expected result; the rings are simply keeping the ball up above the ground, which requires an average force equal and opposite to the gravitational force, mg.]

(b) i. This question is equivalent to: for what $f(x)$ is v_x independent of x_0? If we look at a contact point on the right half, we see that

$$\frac{v_x}{v_y} = -f'(x_0). \tag{16}$$

The distance to the following bounce at $-x_0$ is $-2x_0 = v_x t = v_x(2v_y/g)$. Therefore,

$$v_x v_y = -g x_0. \tag{17}$$

Combining the previous two equations gives

$$v_x = -\sqrt{g x_0 f'(x_0)}. \tag{18}$$

For this to be independent of x_0 we must have

$$f(x) = a\ln(x) + b, \tag{19}$$

where a and b are arbitrary constants (with $a > 0$).

ii. As in part (ii) of part (a), we have $\overline{F}_x = mgv_x/v_y$. Using eq. (16), we obtain $\overline{F} = -mgf'(x_0)$. For this to be independent of x_0 we must have

$$f(x) = cx + d, \tag{20}$$

where c and d are arbitrary constants (with $c > 0$).

5. First, note that it is believable that the system will eventually come to halt, because it loses energy at each pivot point change. At each change, only the component of the mass's velocity that is perpendicular to the new pivot stick survives. The new speed is therefore $\cos\theta$ times the old speed. The energy thus decreases by a factor of $\cos^2\theta$ at each pivot point change.

   Let the angle between the pivot stick and the vertical be ϕ. The component of gravity acting on the mass in the tangential direction around the pivot point is $g\sin\phi \approx g\phi$, for small ϕ. $F = ma$ gives $\ddot{\phi} = (g/\ell)\phi$. The solution to this differential equation is (with $\gamma \equiv \sqrt{g/\ell}$)

$$\phi(t) = A\epsilon^{\gamma t} + B\epsilon^{-\gamma t}, \tag{21}$$

   where A and B are constants determined by the initial conditions. If given initial conditions are $\phi(0) \equiv \phi_0$ and $\dot{\phi}(0) \equiv \omega_0$, then it is easy to solve for A and B, and we obtain

$$\phi(t) = \left(\frac{\gamma\phi_0 + \omega_0}{2\gamma}\right)\epsilon^{\gamma t} + \left(\frac{\gamma\phi_0 - \omega_0}{2\gamma}\right)\epsilon^{-\gamma t}. \tag{22}$$

   Note that $\dot{\phi}(t) = 0$ when

$$t = \frac{1}{2\gamma}\ln\left(\frac{\gamma\phi_0 - \omega_0}{\gamma\phi_0 + \omega_0}\right). \tag{23}$$

   We will find the total time of the movement by adding up the times between successive changes in pivot points. More specifically, if t_0 is the first time the pivot point changes, t_1 is the second time the pivot point changes, etc., then the total time is

$$T - t_0 = \sum_{n=1}^{\infty} t_n - t_{n-1}. \tag{24}$$

The quantity $t_n - t_{n-1}$ is given by twice the t in eq. (23) (because the mass must rise to the point where its speed is zero, and then fall back down), with $\phi_0 = \pm\theta/2$ and ω_0 equal to the angular velocity right after the nth change of pivot point. This angular velocity right after the nth change of pivot point is simply $\cos^n\theta$ times the angular velocity right before the first change in pivot point. The latter can be found by equating the kinetic energy right before the first change in pivot point to the loss in potential energy from the initial balancing point (and using $\cos\beta \approx 1 - \beta^2/2$); we obtain $\omega = \mp\gamma\theta/2$, just before t_0.

Therefore, the angular speed right after the nth pivot point change is equal to $\mp(\cos^n\theta)\gamma\theta/2$, and eq. (23) gives

$$t_n - t_{n-1} = \frac{1}{\gamma}\ln\left(\frac{1+\cos^n\theta}{1-\cos^n\theta}\right). \tag{25}$$

The total time, $T - t_0$, is thus

$$T - t_0 = \sum_{n=1}^{\infty}\frac{1}{\gamma}\ln\left(\frac{1+\cos^n\theta}{1-\cos^n\theta}\right). \tag{26}$$

Now let's try to simplify this, to show the leading behavior in $1/\theta$.

With $x \equiv \cos\theta$, we have, using the two hints,

$$\begin{aligned} T - t_0 &= \frac{1}{\gamma}\left(\sum_{n=1}^{\infty}\ln(1+x^n) - \sum_{n=1}^{\infty}\ln(1-x^n)\right) \\ &= \frac{2}{\gamma}\sum_{n=1}^{\infty}\sum_{k\ \text{odd}}^{\infty}\frac{x^{nk}}{k} \\ &= \frac{2}{\gamma}\sum_{k\ \text{odd}}^{\infty}\frac{1}{k}\frac{x^k}{1-x^k} \qquad \text{(doing the } n \text{ sum first).} \end{aligned} \tag{27}$$

We can now use the approximation $x^k = \cos^k\theta \approx (1-\theta^2/2)^k \approx 1 - k\theta^2/2$, but we must be careful to use this only in its range of validity, $k\theta^2/2 \ll 1$, i.e., $k \ll \theta^2/2$. We'll use the approximation for the range of k up to $M \equiv \eta/\theta^2$, with $\eta \ll 1$. In what follows, we will assume that θ is small enough so that we may choose η to satisfy $\theta \ll \eta \ll 1$. Note that we then have $M \gg 1$.

We may write $T - t_0$ as

$$T - t_0 \approx \frac{2}{\gamma}\sum_{k\ \text{odd}}^{M}\frac{1}{k}\left(\frac{1-k\theta^2/2}{k\theta^2/2}\right) + \frac{2}{\gamma}\sum_{k\ \text{odd},M}^{\infty}\frac{1}{k}\frac{x^k}{1-x^k}. \tag{28}$$

The second term is less than

$$\begin{aligned}
\frac{2}{\gamma}\sum_{k\ \mathrm{odd},M}^{\infty}\frac{x^k}{M(1-x^M)} &= \frac{2}{\gamma}\frac{1}{M(1-x^M)}\frac{x^M}{1-x^2} \\
&\approx \frac{2}{\gamma}\frac{1}{M(M\theta^2/2)}\frac{1}{\theta^2} \\
&= \frac{4}{\gamma\eta^2}. \qquad (29)
\end{aligned}$$

The first term is approximately equal to

$$\begin{aligned}
\frac{4}{\gamma\theta^2}\sum_{k\ \mathrm{odd}}^{M}\frac{1}{k^2} &\approx \frac{4}{\gamma\theta^2}\sum_{k\ \mathrm{odd}}^{\infty}\frac{1}{k^2} \\
&= \frac{\pi^2}{2\gamma\theta^2}. \qquad (30)
\end{aligned}$$

Since we have chosen η to satisfy $\theta \ll \eta \ll 1$, the first term above dominates, and we obtain

$$T - t_0 \approx \sqrt{\frac{\ell}{g}}\left(\frac{\pi^2}{2\theta^2}\right). \qquad (31)$$

6. (a) If the moment of inertia of C is zero, then there can be no torque on C (because otherwise there would be infinite angular acceleration). Hence the rubber band can apply no force to C. Therefore, C moves under the influence of only gravity, just like the block. So they have the same speed at all times (since they started out at the same speed).

   (b) i. Let's first define a few variables. These are all functions of t, but we won't bother writing the t dependence.

      A. Let f be the fraction of the distance C's position is along the rubber band.
      B. Let V_B be the speed of the block (then $V_B = tg\sin\theta \equiv g_\theta t$).
      C. Let v_b be the speed of the band at its point of contact with C. (then $v_b = fV_B = fg_\theta t$.
      D. Let V be the speed of C.
      E. Let ω be the angular velocity of C.
      F. Let a be the acceleration of C.
      G. Let α be the angular acceleration of C.
      H. Let F be the force the band applies to C, defined so that upward along the plane is positive.

Then, since $V = v_b + r\omega$, we have

$$V = fg_\theta t + r\omega. \tag{32}$$

Looking at the net force on C along the plane, we have $mg_\theta - F = ma$. Looking at the torque on C, we have $rF = \rho m r^2 \alpha$. Eliminating F gives $g_\theta - \rho r\alpha = a$. Integrating this from time zero to time t gives (since ω and V are zero at $t = 0$)

$$V = g_\theta t - \rho r\omega. \tag{33}$$

Adding eq. (33) plus ρ times eq. (32) gives

$$V = g_\theta t \left(\frac{1+\rho f}{1+\rho}\right). \tag{34}$$

So the ratio of C's speed to the block's speed is $(1+\rho f)/(1+\rho)$. Now, let f_∞ be the limiting value of f, as $t \to \infty$. Then for large t, ratio of C's speed to the block's speed is the constant $(1+\rho f_\infty)/(1+\rho)$. Therefore, for large t, the ratio of the distances travelled must be this same ratio, i.e.,

$$f_\infty = \frac{1+\rho f_\infty}{1+\rho}. \tag{35}$$

The solution to this equation is $f_\infty = 1$.

ii. Using the fact that the position of the block is $1 + g_\theta t^2/2$, we have, by definition of f,

$$\int V dt = f(1 + g_\theta t^2/2). \tag{36}$$

Differentiating this equation gives

$$V = fg_\theta t + (1 + g_\theta t^2/2)f'. \tag{37}$$

This says nothing more than that C's speed comes partly from the stretching of the band (the $fg_\theta t$ piece) and partly from the movement relative to the band (the term with f'). For large t, we may drop the f' term on the right, compared to the $g_\theta t^2 f'/2$ term. Then, using eq. (34), eq. (37) becomes

$$1 = f + \frac{1+\rho}{2} f' t. \tag{38}$$

If we let the leading behavior of f be $f \approx 1 - ct^{-a}$, the previous equation gives $a = 2/(1+\rho)$. Thus,

$$f \approx 1 - ct^{-2/(1+\rho)}. \tag{39}$$

The relative speed of the block and C is $V_{\rm rel} = V_B - V = g_\theta t - g_\theta t(1+\rho f)/(1+\rho)$. Using the form of f in eq. (39) gives

$$V_{\rm rel} = \frac{g_\theta \rho c}{1+\rho} t^{-\frac{1-\rho}{1+\rho}}. \tag{40}$$

A few examples:

A. If $\rho = 0$, then $V_{\rm rel} = 0$, as we found in part (a).

B. If $\rho = 1/2$, then $V_{\rm rel} \sim t^{-1/3}$, and goes to zero as $t \to \infty$.

C. If $\rho = 1$, then $V_{\rm rel}$ approaches a constant as $t \to \infty$.

D. If $\rho = 2$ (for example, a spool of thread that hangs down below the rubber band), then $V_{\rm rel} \sim t^{1/3}$, and goes to infinity as $t \to \infty$; but this relative speed becomes negligible compared to the block's speed $g_\theta t$, as $t \to \infty$; so f does indeed approach 1, as we found in part (i).

1997
The Boston Area Undergraduate
Physics Competition

April 26, 1997
Time: 4 hours

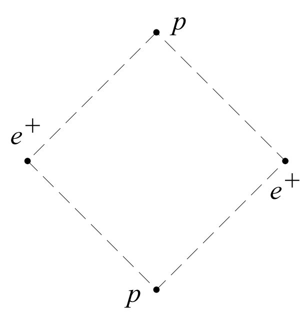
Each of the six questions is worth 10 points.

1. These two unrelated problems are each worth 5 points.

   (a) A block of ice containing a small air bubble floats in a pond. The ice melts. Does the water level go up, go down, or stay the same? Explain.

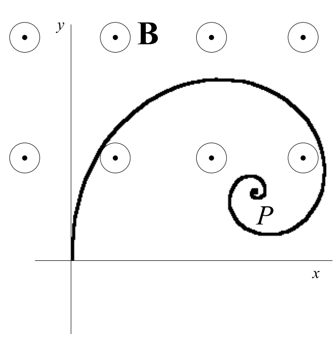
   Answer the same question for the situation where the air bubble is replaced by a small piece of lead.

   (b) Two protons are held at opposite corners of a square, and two positrons are held at the other corners. The square has sides of length $\ell = 10^{-3}$ m.



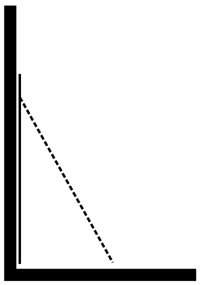
   The particles are released. Give an approximate value (one significant figure is sufficient) for the speeds of the particles when they are very far apart.
   Notes: The mass of a positron is $m \approx 9.1 \cdot 10^{-31}$ kg, and the mass of a proton is $M \approx 1.7 \cdot 10^{-27}$ kg. Coulomb's Law says that the force between any two of these particles, separated by r meters, is equal to $2.3 \cdot 10^{-28}/r^2$ Newtons.

2. Let $\mathcal{R}_{y>0}$ be the region of space where $y > 0$. Let $\mathcal{R}_{y>0}$ contain a constant magnetic field $\vec{B} = B\hat{z}$ (see the figure below; $\vec{B}$ points out of the page).



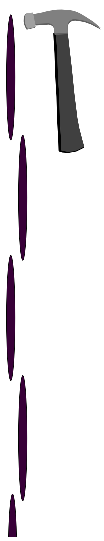
   A particle with charge q and mass m travels along the y-axis and enters $\mathcal{R}_{y>0}$ with speed v_0.
   Assume that in $\mathcal{R}_{y>0}$ the particle is subject to a friction force, $\vec{F}_f$, proportional to its velocity, i.e., $\vec{F}_f = -\alpha\vec{v}$. (You may ignore gravity.)
   Assume this friction force is large enough so that the particle will remain inside $\mathcal{R}_{y>0}$ at all times. The particle will then spiral in toward the point P. What are the coordinates of the point P?

3. A stick of length ℓ and uniform mass density per unit length leans against a frictionless wall. The ground is also frictionless.



The stick is initially held motionless, with its bottom end an infinitesimal distance from the wall. The stick is then released, whereupon the bottom end begins to slide away from the wall, and the top end begins to slide down the wall.
A long time after the stick is released, what is the horizontal component of the velocity of its center of mass?

4. This problem deals with rigid 'stick-like' objects of length $2r$, masses M_i, and moments of inertia $\rho M_i r^2$, where ρ is a numerical constant.

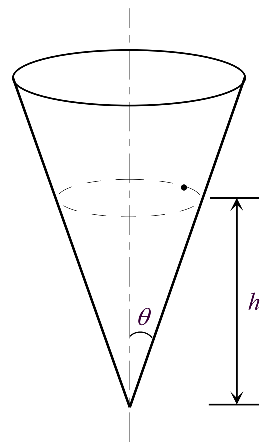


The center of mass of each stick is located at the center of the stick. (All the sticks have the same r, ρ, and relative mass distribution. Only the masses differ.) Assume $M_1 \gg M_2 \gg M_3 \gg \cdots$.
The sticks are placed on a horizontal frictionless surface. The ends overlap a negligible distance, and the ends are a negligible distance apart.
The first (heaviest) stick is given an instantaneous blow (as shown) which causes it to translate and rotate. (The blow comes from the side of stick #1 on which stick #2 lies [the right side, as shown in the figure].) Depending on the value of ρ, the first stick may strike the second stick, which will then strike the third stick, and so on. Assume all collisions among the sticks are elastic.
Depending on the value of ρ, the speed of the nth stick will either (1) approach zero, (2) approach infinity, or (3) be independent of n, as $n \to \infty$.

What is the special value of ρ corresponding to the third of these three scenarios? Give an example of a stick having this value of ρ.

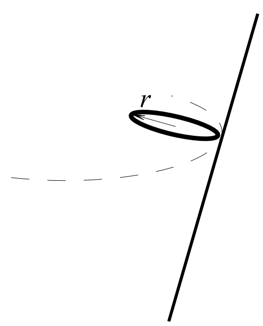
Note: You may work in the approximation where M_1 is infinitely heavier than M_2, which is infinitely heavier than M_3, etc.

5. (a) (2 points) A fixed cone stands on its tip, with its axis in the vertical direction.



When viewed from the side, the cone subtends an angle of 2θ. A particle of negligible size slides on the inside surface of the cone. This surface is frictionless.
Assume conditions have been set up so that the particle moves in a circle at a height h above the tip.
What is the frequency, ω, of this circular motion?

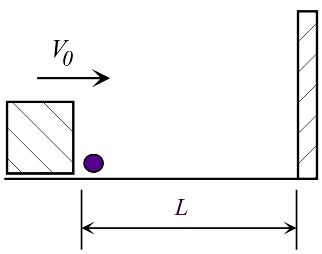
(b) (8 points) Assume now that the surface has friction, and a small ring of radius r rolls on the surface without slipping.



Assume conditions have been set up so that (1) the point of contact between the ring and the cone moves in a circle at a height h above the tip, and (2) the plane of the ring is at all times perpendicular to the line joining the point of contact and the tip of the cone.
What is the frequency, ω, of this circular motion? How does it compare to the answer in part (a)?

Note: You may work in the approximation where r is much smaller than the radius of the circular motion, $h \tan\theta$.

6. A block with very large mass M slides on a frictionless surface towards a fixed wall. The block's speed is V_0. The block strikes a particle with very small mass m (and negligible size), which is initially at rest at a distance L from the wall. The particle bounces elastically off the block and slides to the wall, where it bounces elastically and then slides back toward the block. The particle continues to bounce elastically back and forth between the block and the wall.



(a) (7 points) How close does the block come to the wall?

(b) (3 points) How many times does the particle bounce off the block, by the time the block makes its closest approach to the wall?

Note: In both parts (a) and (b), you may assume $M \gg m$, and you need only obtain *approximate answers, valid to leading order in* m/M. (In other words, pick quantities of, say, $M = 10$ kg, $m = 1$ g, $L = 1$ m, and $V_0 = 1$ m/s, and obtain approximate answers to the above two questions.)

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Physics Competition

Solutions

1. (a) In case of the air bubble the level will stay the same, since the volume of the displaced water is exactly equal to the "extra" volume of water that came from the molten block.

   In case of the frozen in piece of lead, the level of water will decrease, because the volume of the water displaced due to the lead piece exceeds the volume of the piece.

   (b) Since the positrons are a lot lighter than the protons, it makes sense to divide the problem in two parts.

   First, the positrons fly away while the protons stay in place. The energy conservation requires that

$$4 \times \frac{1}{2}\alpha(\frac{1}{l} + \frac{1}{l} + \frac{1}{\sqrt{2}l}) = \frac{\alpha}{\sqrt{2}l} + 2\frac{mv_{e^+}^2}{2}, \tag{1}$$

   where $\alpha = 2.3 \times 10^{-28} Jm$, or

$$v_{e^+} = \sqrt{\frac{\alpha}{ml}(4 + 1/\sqrt{2})} \approx 1100 m/s \tag{2}$$

   After that, the protons fly away:

$$v_p = \sqrt{\alpha/\sqrt{2}Ml} \approx 9.8 m/s. \tag{3}$$

2. The forces on the particle are the friction force and the magnetic force, so $\vec{F} = -\alpha\vec{v} + q\vec{v} \times \vec{B}$. The particle will remain in the x-y plane, so $\vec{v}$ has no component in the $\hat{z}$ direction. The cross product takes a simple form, and we have

$$\begin{aligned} F_x &= -\alpha v_x + qBv_y, \\ F_y &= -\alpha v_y - qBv_x. \end{aligned} \tag{4}$$

   Equivalently,

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -\alpha\frac{dx}{dt} + qB\frac{dy}{dt}, \\ m\frac{d^2y}{dt^2} &= -\alpha\frac{dy}{dt} - qB\frac{dx}{dt}. \end{aligned} \tag{5}$$

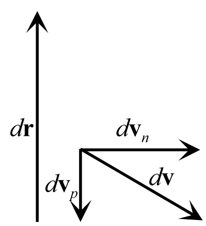
Integrating these equations from the time the particle enters $\mathcal{R}_{y>0}$ to the time it comes to rest, we obtain

$$\begin{aligned} m\Delta v_x &= -\alpha\Delta x + qB\Delta y, \\ m\Delta v_y &= -\alpha\Delta y - qB\Delta x, \end{aligned} \tag{6}$$

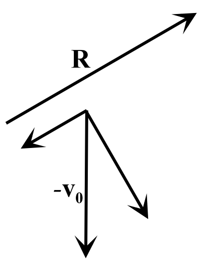
We want to find the coordinates of P, namely $(\Delta x, \Delta y)$.

The initial velocity is $\vec{v} = (0, v_0, 0)$, and the final velocity is $\vec{v} = (0, 0, 0)$, so $\Delta v_x = 0$ and $\Delta v_y = -v_0$. Solving eqs. (6) for Δx and Δy gives

$$\Delta x = \frac{qBmv_0}{\alpha^2 + (qB)^2}, \qquad \Delta y = \frac{\alpha m v_0}{\alpha^2 + (qB)^2}. \tag{7}$$



There is an equivalent geometrical solution. Consider a small interval of time dt. The particle will change its position by $d\mathbf{r} = \mathbf{v}dt$. Over the same time period, the speed will change by $d\mathbf{v}_p = \frac{\alpha}{m}vdt$ opposite to the direction of motion, and by $d\mathbf{v}_n = \frac{qB}{m}vdt$ in the direction normal to it. Note that the components of the velocity change are proportional to the displacement.



Similar set of vectors can be drawn for any small dt. Summing them up, and knowing the total change in the particle speed $(-\mathbf{v}_0)$, we can draw the final diagram. Since the proportionality is conserved in vector addition, the total displacement $\mathbf{R}$ (and, of course, its components) can be easily found.

3. The key to this problem is to realize that the stick will lose contact with the wall before it hits the ground. The first thing we must do is calculate exactly where this loss of contact occurs.

   Let $r = \ell/2$, for convenience. It is easy to see that while the stick is in contact with the wall, the center of mass of the stick will move in a circle of radius r. Let θ be the angle between the wall and the radius from the corner to the CM of the stick. (This is also the angle between the stick and the wall.)

   We will solve the problem by assuming that the CM always moves in a circle, and then determining the point at which the horizontal CM speed starts to decrease (i.e., the point at which the normal force from the wall becomes negative [which it of course can't do]).

By conservation of energy, the kinetic energy of the stick is equal to the loss in potential energy, which is $mgr(1-\cos\theta)$, where θ is defined above. This kinetic energy may be broken up into the CM translational energy plus the rotation energy. The CM translational energy is simply $mr^2\dot{\theta}^2/2$ (since the CM travels in a circle). The rotational energy is $I\dot{\theta}^2/2$. (The same $\dot{\theta}$ applies here as in the CM translational motion, because θ is the angle between the stick and the vertical.) Letting $I \equiv \rho mr^2$, to be general ($\rho = 1/3$ for our stick), we have, by conservation of energy, $(1+\rho)mr^2\dot{\theta}^2/2 = mgr(1-\cos\theta)$. Therefore, the speed of the CM, $v = r\dot{\theta}$, is

$$v = \sqrt{\frac{2gr}{1+\rho}}\sqrt{(1-\cos\theta)}\,. \tag{8}$$

The horizontal speed is therefore

$$v_x = \sqrt{\frac{2gr}{1+\rho}}\sqrt{(1-\cos\theta)}\cos\theta. \tag{9}$$

Taking the derivative of $\sqrt{(1-\cos\theta)}\cos\theta$, we see that the speed is maximum at $\cos\theta = 2/3$. (This is independent of ρ.)

Therefore the stick loses contact with the wall when

$$\cos\theta = 2/3. \tag{10}$$

Using this value of θ in eq. (9) gives a horizontal speed of (letting $\rho = 1/3$)

$$v_x = \frac{1}{3}\sqrt{2gr} = \frac{1}{3}\sqrt{gl}. \tag{11}$$

This is the horizontal speed just after the stick loses contact with the wall, and thus is the horizontal speed from then on, because the floor exerts no horizontal force.

4. Consider the collision between two sticks. Let the speed of the end of the heavy one be V. Since this stick is essentially infinitely heavy, we may consider it to be an infinitely heavy ball, moving at speed V. (The translational degree of freedom of the heavy stick is irrelevant, as far as the light stick is concerned.)

   In the same spirit as the (easier) problem of the collision between two balls of greatly disparate masses, we will work out this problem in the rest frame of the infinitely heavy ball right before the collision. (The problem can be done in the lab frame, but our method here is a little less messy.) The situation reduces to a stick of mass m, length $2r$, moment of inertia ρmr^2, and speed V, approaching a fixed wall To find the behavior of the stick after the collision, we will use (1) conservation of energy, and (2) conservation of angular momentum around the contact point.

Let u be the speed of the center of mass of the stick after the collision. Let ω be its angular velocity after the collision.

Since the wall is infinitely heavy, it will acquire zero kinetic energy. So conservation of E gives

$$\frac{1}{2}mV^2 = \frac{1}{2}mu^2 + \frac{1}{2}(\rho m r^2)\omega^2. \tag{12}$$

The initial angular momentum around the contact point is $L = mrV$, so conservation of L gives (breaking L after the collision up into the L of the CM plus the L relative to the CM)

$$mrV = mru + (\rho m r^2)\omega. \tag{13}$$

Solving eqs. (12) and (13) for u and $r\omega$ in terms of V gives

$$u = V\frac{1-\rho}{1+\rho}, \qquad \text{and} \qquad rw = V\frac{2}{1+\rho}. \tag{14}$$

(The other solution, $u = V$ and $r\omega = 0$ represents the case where the stick misses the wall.)

Going back to the lab frame (i.e., subtracting V from the speed u) we see that the collision gives the the lighter stick a CM speed equal to $v = 2V\rho/(1+\rho)$ in the same direction as the original V. But the far end of the light stick has a backwards rotational speed equal to $r\omega = 2V/(1+\rho)$. This rotational speed is greater than the CM speed, so the far end of the light stick travels at a speed

$$V' = r\omega - v = V\frac{2(1-\rho)}{1+\rho} \tag{15}$$

in the direction opposite to the original V.

The same analysis works in the next collision. In other words, the bottom ends of the sticks move with speeds that form a geometric progression with ratio $2(1-\rho)/(1+\rho)$. If this ratio is less than 1 (i.e., $\rho > 1/3$), then the speeds go to zero, as $n \to \infty$. If it is greater than 1 (i.e., $\rho < 1/3$), then the speeds go to infinity, as $n \to \infty$. If it equals 1 (i.e., $\rho = 1/3$), then the speeds are independent of n, as $n \to \infty$.

Therefore,

$$\rho = \frac{1}{3} \tag{16}$$

is the desired answer.

A uniform stick has $\rho = 1/3$ (usually written in the form $I = m\ell^2/12$, where $\ell = 2r$).

Although in the case where ρ is strictly equal to zero the centers of masses of the sticks would remain at rest while their rotational speeds would form a geometric progression with ratio 2, we are not considering this to be the correct answer. Real objects with finite mass have a finite moment of inertia. A proper physical approach then would require taking a limit as $\rho \to 0$, but in that case the limit as $n \to \infty$ is infinite.

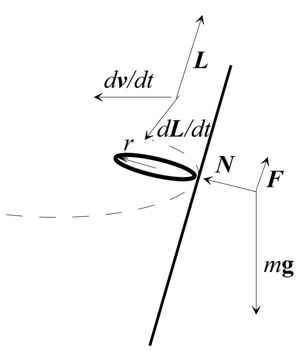
5. (a) The forces on the particle are gravity (mg) and the normal force (N) from the cone. In our situation, there is no net force in the vertical direction, so

$$N \sin\theta = mg, \tag{17}$$

i.e., $N = mg/\sin\theta$. Therefore, the inward horizontal force, $N\cos\theta$, equals $mg/\tan\theta$. This force must account for the centripetal acceleration of the particle moving in a circle of radius $h\tan\theta$. Hence, $mg/\tan\theta = m(h\tan\theta)\omega^2$, and

$$\omega = \sqrt{\frac{g}{h}\frac{1}{\tan\theta}}. \tag{18}$$

(b)



The forces on the ring are gravity (mg), the normal force (N) from the cone, and a friction force (F) pointing up along the cone. In our situation, there is no net force in the vertical direction, so

$$N\sin\theta + F\cos\theta = mg. \tag{19}$$

The fact that the inward horizontal force accounts for the centripetal acceleration yields

$$N\cos\theta - F\sin\theta = m(h\tan\theta)\omega^2. \tag{20}$$

We must now consider the torque, $\vec{\tau}$, on the ring. The torque is due solely to F (because gravity provides no torque, and N points though the center of the ring, by assumption (2) in the problem). So

$$\tau = rF, \tag{21}$$

and it points in the direction along the circular motion. Since $\vec{\tau} = d\vec{L}/dt$, we must now find $d\vec{L}/dt$.

$\vec{L}$ is made up of two pieces. One comes from the center of mass motion of the ring, which revolves around the axis of the cone. This part of $\vec{L}$ does not change, so we may neglect it in calculating $d\vec{L}/dt$. The other piece comes from the rotation of the ring. Let us call this part $\vec{L}'$. It points up along the cone, so the $\vec{L}'$ vector traces out a cone in which the tip of $\vec{L}'$ moves in a circle of radius $L'\sin\theta$. The frequency of this circular motion is of course the same ω as above. Therefore,

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}'}{dt} = \omega L' \sin\theta \tag{22}$$

(in the direction of the circular motion).
Thus, $\vec{\tau} = d\vec{L}/dt$ gives

$$rF = \omega L' \sin\theta. \tag{23}$$

But $L' = mr^2\omega'$, where ω' is the angular speed of the ring. And we know that ω' and ω are related by $r\omega' = (h\tan\theta)\omega$ (the rolling-without-slipping condition)[1]. Therefore $L' = mr(h\tan\theta)\omega$. Using this in eq. (23) yields

$$F = m\omega^2(h\tan\theta)\sin\theta. \tag{24}$$

Eqs. (19), (20), and (24) have the three unknowns, N, F, and ω. We can solve for ω by multiplying eq. (19) by $\cos\theta$, and eq. (20) by $\sin\theta$, and taking the difference, to obtain

$$F = mg\cos\theta - m\omega^2(h\tan\theta)\sin\theta. \tag{25}$$

Equating this expression for F with that in eq. (24) gives

$$\omega = \sqrt{\frac{g}{2h}\frac{1}{\tan\theta}}. \tag{26}$$

This frequency is $1/\sqrt{2}$ times the frequency found in part (a).
REMARK: If one considers an object with moment of inertia ρmr^2 (our ring has $\rho = 1$), then one can show by the above reasoning that the "2" in eq. (26) is simply replaced by $(1+\rho)$.

6. (a) **First Solution:**
Consider one of the collisions of the block and the particle. Let V and v be the speeds of the block and particle, respectively, after the collision. Let the collision occur at a distance ℓ from the wall. We claim the the quantity $\ell(v - V)$ is an invariant (i.e., for each collision, it is the same). The proof is as follows.

[1]Actually, this isn't quite true, for the same reason that the earth spins around 366 instead of 365 times in a year. But it's valid enough, in the limit of small r.

Let us find the position, ℓ', of the following collision. The time to the next collision is given by $Vt + vt = 2\ell$ (because the sum of the distances traveled by the two objects is 2ℓ). Therefore, since $\ell' = \ell - Vt$, we have

$$\ell' = \ell(v - V)/(v + V). \tag{27}$$

We now invoke the handy fact that in an elastic collision, the relative speed of the particles before the collision is equal to the relative speed after the collision. (This is most easily proved by working in the center of mass frame.) Therefore, letting V' and v' be the speeds of the block and particle, respectively, after the next collision, we have

$$v + V = v' - V'. \tag{28}$$

Using this in eq. (27) gives

$$\ell'(v' - V') = \ell(v - V), \tag{29}$$

as was to be shown.

What is the value of this invariant? After the first collision, the block continues to move essentially with speed V_0 (up to corrections of order m/M), and the particle acquires a speed essentially equal to $2V_0$ (up to corrections of order m/M). (The latter is most easily seen by working in the frame of the heavy block.) So the invariant $\ell(v - V)$ is essentially equal to $L(2V_0 - V_0) = LV_0$.

Let L_c be the closest distance to the wall. When the block is at this closest point to the wall, its speed is zero. Therefore, all of the initial kinetic energy of the block belongs to the particle. Thus, $v = V_0\sqrt{M/m}$. So at this point our invariant tells us $LV_0 \approx L_c(V_0\sqrt{M/m} - 0)$, and so

$$L_c \approx L\sqrt{\frac{m}{M}}. \tag{30}$$

Second Solution:

Let $V(t)$ be the speed of the block, and let $v(t)$ be the speed of the particle. Let $x(t)$ be the distance from the wall to the block.

The block reaches its closest point to the wall when all of its initial kinetic energy is transferred to the particle, i.e.,

$$\frac{1}{2}mv^2 = \frac{1}{2}MV_0^2. \tag{31}$$

Hence, $v = V_0\sqrt{M/m}$ at this point. Therefore, if we can find a relation between $v(t)$ and $x(t)$, we are done.

We claim that for $M \gg m$, the product $v(t)x(t)$ is essentially equal to $V_0 L$. The proof is as follows.

Consider the later times when the bounces are very frequent, and when v is very large, so that a large number, dn, of bounces occur during a small period of time, dt, where x does not change significantly.

Since the particle travels a distance $v\,dt$ during a time dt, and since the distance from the block to the wall and back is $2x$, we have

$$dn = \frac{v\,dt}{2x}. \tag{32}$$

Each collision between the block and the particle increases the particle's speed by essentially $2V$ (because $M \gg m$, so the block is essentially infinitely heavy). Therefore, the increase in the speed of the particle during the time dt is

$$dv = 2V\,dn = \frac{V v\,dt}{x}. \tag{33}$$

But $V = -dx/dt$, so we have

$$dv = -\frac{v\,dx}{x}. \tag{34}$$

Dividing by v and integrating gives $\ln v = -\ln x + (\text{const})$. Therefore,

$$vx = C. \tag{35}$$

Thus, the speed of the particle is inversely proportional to the distance between the wall and the block.

This is just what we expect if we consider the particle to be a one-dimensional gas. Indeed, the adiabatic compression of such gas would be governed by an invariant $PV^\gamma = const$, where

$$\gamma = \frac{C_P}{C_V} = \frac{C_V + 1}{C_V} = \frac{\frac{i}{2}+1}{\frac{i}{2}}, \tag{36}$$

where i is the number of degrees of freedom. For the one-dimensional gas, $i = 1$, so $\gamma = 3$ and $PV^3 = const$. Since PV is proportional to T, or v^2, V is x in the one-dimensional case, $v^2x^2 = const$, or $vx = C$.

We must now determine the constant C by looking at the first few collisions.

The first collision gives the particle a speed $2V_0$ (it's not quite $2V_0$, but the error is of order m/M). After the collision, the block continues to move with (essentially) speed V_0, so it is easy to calculate that the second collision occurs at (roughly) $x = L/3$.

After the second collision, the particle has speed $4V_0$,[2] while the block continues to have speed V_0, so we find that the third collision occurs at $x = L/5$.

After the third collision, the particle has speed $6V_0$, while the block continues to have speed V_0, so we find that the fourth collision occurs at $x = L/7$.

In general, we find that the kth collision occurs at $L/(2k-1)$. And the particle had speed $2(k-1)V_0$ before the collision, and $2kV_0$ after it. This is valid up to order m/M corrections, as long as k is not too large.

For large enough M/m, we may make this realm overlap with the above realm where there is a large number of collisions in a short period of time. So we find $C = vx \approx (2kV_0)L/(2k-1) \approx V_0 L$, since k can be made large if M/m is very large. Therefore,

$$vx \approx V_0 L. \tag{37}$$

Using $v \approx V_0 L/x$ in eq. (31), we find that the closest approach, x_c, of the block to the wall is

$$x_c \approx L\sqrt{\frac{m}{M}}. \tag{38}$$

(b) **First Solution:**

Let V and v be the speeds of the block and particle, respectively.

The decrease in the momentum of the block due to a bounce is equal to the change in the momentum of the particle from the bounce, which is roughly equal to $2mv$ (we are assuming V small compared to v, which is the case after the first few collisions). If there are dn bounces in a time dt, then conservation of momentum during the time dt gives (assuming v stays fairly constant throughout the small interval dt).

$$MdV = -2mvdn. \tag{39}$$

Conservation of energy, $MV^2/2 + mv^2/2 = MV_0^2/2$, allows us to write v in terms of V:

$$v = V_0\sqrt{\frac{M}{m}}\sqrt{1-\frac{V^2}{V_0^2}}. \tag{40}$$

Eq. (39) then gives (changing variables to $y \equiv V/V_0$, and integrating up to the

[2]It should be understood that this and all the other numbers in the next few paragraphs are approximations which become arbitrarily accurate in the limit $M \gg m$.

closest approach to the wall, which corresponds to $V = 0$, and hence $y = 0$)

$$\frac{1}{2}\sqrt{\frac{M}{m}}\int_1^0 \frac{dy}{\sqrt{1-y^2}} = -\int_0^N dn = -N. \tag{41}$$

(The integrand is actually not valid for V near V_0, i.e. for y near 1, because eq. (39) is not valid there. But the error there is small compared to the total number of bounces.)

The integral gives $\arcsin y$, which yields $-\pi/2$ when evaluated between 1 and 0. So the total number of bounces is

$$N \approx \frac{\pi}{4}\sqrt{\frac{M}{m}}, \tag{42}$$

Second Solution:

Let V and v be the speeds of the block and particle, respectively, after a given collision. Let V' and u' be the speeds of the block and particle, respectively, after the following collision. Conservation of momentum in this second collision gives

$$MV - mv = MV' + mv'. \tag{43}$$

This equation, together with eq. (28), allows us to solve for V' and v' in terms of V and v. In matrix form, we obtain

$$\begin{pmatrix} V' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{M-m}{M+m} & \frac{-2m}{M+m} \\ \frac{2M}{M+m} & \frac{M-m}{M+m} \end{pmatrix} \begin{pmatrix} V \\ v \end{pmatrix}. \tag{44}$$

The eigenvectors, A_i, and eigenvalues, λ_i, of this matrix are

$$A_1 = \begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix}, \quad \lambda_1 = \frac{M-m}{M+m} + \frac{2i\sqrt{Mm}}{M+m} \equiv e^{i\theta}, \tag{45}$$

$$A_2 = \begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix}, \quad \lambda_2 = \frac{M-m}{M+m} - \frac{2i\sqrt{Mm}}{M+m} \equiv e^{-i\theta}, \tag{46}$$

where $\theta \equiv \arctan(2\sqrt{Mm}/(M+m)) \approx 2\sqrt{m/M}$.

The initial conditions are $(V, v) = (V_0, 0) = (V_0/2)(A_1 + A_2)$. Therefore, the speeds after the nth bounce are given by

$$\begin{pmatrix} V_n \\ v_n \end{pmatrix} = \frac{V_0}{2}(\lambda_1^n A_1 + \lambda_2^n A_2). \tag{47}$$

Writing $\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$, and using the explicite form of the A_i, we have

$$\begin{pmatrix} V_n \\ v_n \end{pmatrix} = \frac{V_0}{2}(e^{in\theta}A_1 + e^{-in\theta}A_2) = V_0 \begin{pmatrix} \cos n\theta \\ \sqrt{\frac{M}{m}}\sin n\theta \end{pmatrix}. \qquad (48)$$

The block makes its closest approach to the wall when $V_N = 0$, i.e., when $N\theta = \pi/2$. Using the definition of θ gives

$$\begin{aligned} N &= \frac{\pi}{2}\frac{1}{\arctan\frac{2\sqrt{Mm}}{M+m}} \\ &\approx \frac{\pi}{4}\sqrt{\frac{M}{m}}. \end{aligned} \qquad (49)$$

The Boston Area Undergraduate Physics Competition

April 18, 1998

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Year: ______________________________

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Do not turn this page until you are told to do so.

You have four (4) hours to complete this exam.

Please provide the information requested on this cover sheet. At the end of the exam, hand in this cover sheet with your solutions. You may keep the exam questions.

Show all relevant work in your exam books. Please write neatly. Partial credit will be given for significant progress made toward a correct solution.

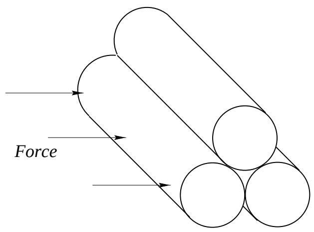
You must be enrolled in a full-time undergraduate program to be eligible for prizes.

1998
Boston Area Undergraduate
Physics Competition

April 18, 1998
Time: 4 hours

Each of the six questions is worth 10 points.

1. Three identical cylinders are arranged in a triangle as shown, with the bottom two lying on the ground.

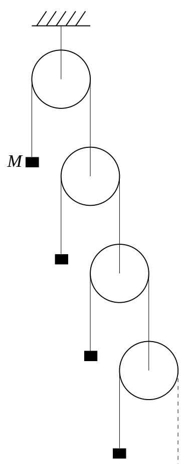


The ground and the cylinders are frictionless. You apply a force (directed to the right) on the left cylinder.
What are the minimum and maximum accelerations you may give to the system in order for all three cylinders to remain in contact with each other?

2. A charged particle sits at the center of a circular region containing a magnetic field, $\vec{B}$. The field $\vec{B} = \vec{B}(r)$ depends only on the radial position r, and it is perpendicular to the plane of the circle. The total magnetic flux through the circle is zero.

   The particle is given a kick. Show that if the particle leaves the circular region, then at the instant it leaves, its velocity points in the radial direction.

3. Consider the infinite Atwood's machine shown in the figure.



A string passes over each pulley, with one end attached to a mass and the other end attached to another pulley. All the masses are equal to M, and all the pulleys and strings are massless.

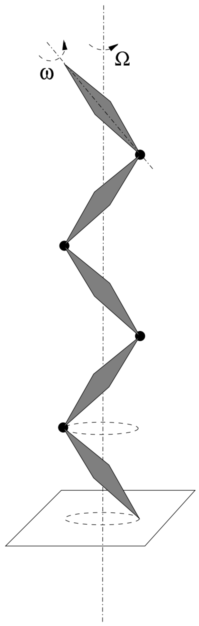
The masses are held fixed and then simultaneously released. What is the acceleration of the top mass?

(You may define this infinite system as follows. Consider it to be made of N pulleys, with a non-zero mass replacing what would have been the $(N + 1)$st pulley. Then take the limit as $N \to \infty$. It is not necessary, however, to use this exact definition.)

4. Each edge of an icosahedron is a 1Ω resistor. Find the effective resistance between two adjacent vertices.

   (An icosahedron consists of 20 equilateral triangles. It has 12 vertices and 30 edges, with 5 edges meeting at each vertex.)

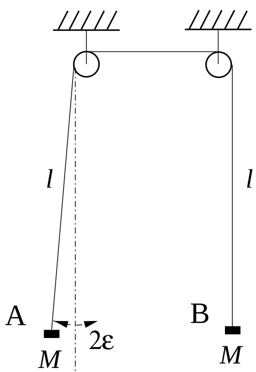
5. Consider the setup of N identical cylindrically symmetric tops in the figure.



The bottom one rests on a frictionless table. Each top is connected to the one above it by a free pivot. The inclination angles of the tops are the same. The center of mass of each top is at the midpoint of its symmetry axis.

You wish to set up a very slow circular precession of the tops, where the CM of each top stays fixed while the ends travel in circles. The angular speed of the top top is ω. Find the angular speeds of all the other tops as functions of ω. (You may work in the approximation where these speeds are very large.)

6. Two masses, A and B, each have mass M and are attached to the ends of a massless string.



The string passes over a set of massless pulleys of negligible size. The masses are at rest at a distance ℓ from the pulleys.

Mass A is then given a very small horizontal kick, so that it initially swings back and forth with amplitude ϵ (where $\epsilon \ll \ell$).

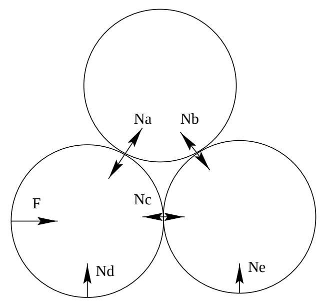
It turns out that after a very long time, one of the masses will eventually rise up and hit its pulley.

(a) (3 points) Which mass hits its pulley?

(b) (7 points) What is the speed of mass B, right before the hitting of the pulley occurs?

(Throughout this problem, work in the approximation where $\epsilon \ll \ell$.)

1. Apart from gravity, the forces on the cylinders include the applied force of magnitude F, normal forces between the bottom two cylinders and the ground with magnitudes N_d and N_e, and three pairs of forces with magnitudes N_a, N_b and N_c between the cylinders (as shown on the diagram).



- The low limit on the acceleration is given by the condidtion that the two bottom cylinders lose contact, i.e., $N_c = 0$. Then, the horizontal component of the force on the right cylinder is

$$Ma = N_b \sin(\pi/6) = N_b/2 \tag{1}$$

The vertical components of the forces on the top cylinder:

$$Mg = N_b \cos(\pi/6) + N_a \cos(\pi/6) \tag{2}$$

and the left cylinder:

$$N_a \sin(\pi/6) = N_a/2 = F - Ma = 2Ma, \tag{3}$$

Eliminating N_a and N_b

$$Mg = (2Ma + 4Ma)\sqrt{3}/2 \tag{4}$$

and $a_{min} = \frac{g}{3\sqrt{3}}$

- When the acceleration is increased beyond a certain value, the top cylinder will lose contact with the right cylinder. It corresponds to vanishing N_b. In this case for the top cylinder the vertical components of the forces:

$$Mg = N_a \cos(\pi/6) \tag{5}$$

For horizontal components we get

$$Ma = N_a \sin(\pi/6) = N_a/2, \tag{6}$$

and $a_{max} = \frac{1}{2}2g/\sqrt{3} = g/\sqrt{3} = 3a_{min}$

2. The force on the particle in the magnetic field is $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \vec{B}$. In cylindrical coordinates, where $\mathbf{r} = r\hat{\mathbf{r}}$, $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}$, and $\ddot{\mathbf{r}} = \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{\mathbf{r}}$, at any given time the tangential projection of the force on the particle is

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = q\dot{r}B \tag{7}$$

By multiplying both sides of the equation by r, we get

$$2mr\dot{r}\dot{\theta} + r^2\ddot{\theta} = qr\dot{r}B, \tag{8}$$

or, extracting the full time differential on the left,

$$\frac{md(r^2\dot{\theta})}{dt} = \frac{qB(r)rdr}{dt}. \tag{9}$$

which could also be seen as the rate of change of angular momentum of the particle $L = mr^2\dot{\theta}$ being equal to the torque about the center of the circular region. Integrating (9) over the time it takes for the particle to exit the circular region,

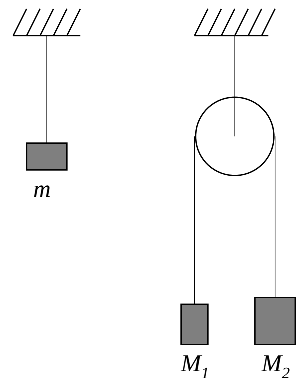
$$r^2\dot{\theta}|_{t,r=0}^{t=T,r=R} = \int_{r=0}^{r=R} qBrdr \tag{10}$$

The right side is $q/2\pi$ times the total flux through the circular region, which is zero, so $\dot{\theta}$ at the time the particle leaves the region must be zero. Since the total velocity of the particle is non-zero (a static magnetic field does not do work), the velocity of the particle then must point radially outward.

3. *First Solution:*

Consider the following auxiliary

Problem: Two set-ups are shown in the figure.



(The first contains a hanging mass m. The second contains a hanging pulley, over which two masses, M_1 and M_2, hang.) Both supports have acceleration a_s downward. What should m be, in terms of M_1 and M_2, so that the tension in the top string is the same in both cases?

Answer: In the first case, we have

$$mg - T = ma_s. \tag{11}$$

In the second case, let a be the acceleration of M_2 relative to the support (with downward taken to be positive). Then we have

$$\begin{aligned} M_1 g - \frac{T}{2} &= M_1(a_s - a), \\ M_2 g - \frac{T}{2} &= M_2(a_s + a). \end{aligned} \tag{12}$$

Note that if we define $g' \equiv g - a_s$, then we may write these three equations as

$$\begin{aligned} mg' &= T, \\ M_1 g' &= \frac{T}{2} - M_1 a, \\ M_2 g' &= \frac{T}{2} + M_2 a. \end{aligned} \tag{13}$$

The last two give $4M_1M_2g' = (M_1 + M_2)T$. The first equation then gives

$$m = \frac{4M_1M_2}{M_1 + M_2}. \tag{14}$$

Note that the value of a_s is irrelevant. (We effectively have a fixed support in a world where the acceleration from gravity is g'.) This problem shows that the two-mass system in the second case may be equivalently treated as a mass m, as far as the upper string is concerned. ■

Now let's look at our infinite Atwood machine. Start at the bottom. (Assume the system has N pulleys, where $N \to \infty$.) Let the bottom mass be x. Then the above problem shows that the bottom two masses, M and x, may be treated as an effective mass $f(x)$, where

$$f(x) = \frac{4x}{1 + (x/M)}. \tag{15}$$

We may then treat the combination of the mass $f(x)$ and the next M as an effective mass $f(f(x))$. These iterations may be repeated, until we finally have a mass M and a mass $f^{(N-1)}(x)$ hanging over the top pulley.

We must determine the behavior of $f^N(x)$, as $N \to \infty$. The behavior is obvious by looking at a plot of $f(x)$ (which we'll let the reader draw). (Note that $x = 3M$ is a fixed point of f, i.e., $f(3M) = 3M$.) It is clear that no matter what x we start with, the iterations approach $3M$ (unless, of course, $x = 0$). So our infinite Atwood machine is equivalent to (as far as the top mass is concerned) just the two masses M and $3M$.

We then easily find that the acceleration of the top mass is (net downward force)/(total mass) $= 2Mg/(4M) = g/2$.

NOTE: As far as the support is concerned, the whole apparatus is equivalent to a mass $3M$. So $3Mg$ is the weight the support holds up.

Second Solution:

Note that if the gravity in the world were multiplied by a factor η, then the tension in all the strings would likewise be multiplied by η. (The only way to make a tension, i.e., a force, is to multiply a mass times g.) Conversely, if we put the apparatus on another planet and discover that all the tensions are multiplied by η, then we know the gravity there must be ηg.

Let the tension in the string above the first pulley be T. Then the tension in the string above the second pulley is $T/2$ (since the pulleys are massless). Let the acceleration of the second pulley be a_{p2}. Then the second pulley effectively lives in a world where the gravity is $g - a_{p2}$. If we imagine holding the string above the second pulley and accelerating downward at a_{p2} (so that our hand is at the origin of the new world), then we really haven't changed anything, so the tension in this string in the new world is still $T/2$.

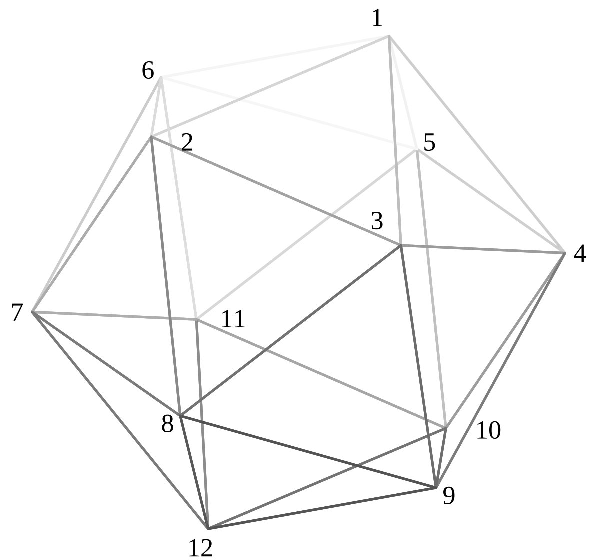
But in this infinite setup, the system of all the pulleys except the top one is the same as the original system of all the pulleys. Therefore, by the arguments in the first paragraph, we must have

$$\frac{T}{g} = \frac{T/2}{g - a_{p2}}. \tag{16}$$

Hence, $a_{p2} = g/2$. (Likewise, the relative acceleration of the second and third pulleys is $g/4$, etc.) But a_{p2} is also the acceleration of the top mass. So our answer is $g/2$.

Note that $T = 0$ also makes eq. (16) true. But this corresponds to putting a mass of zero at the end of a finite pulley system.

4. This problem is just another variation of the popular resistor cube problem, offered, among other places, in BAUPC'96.



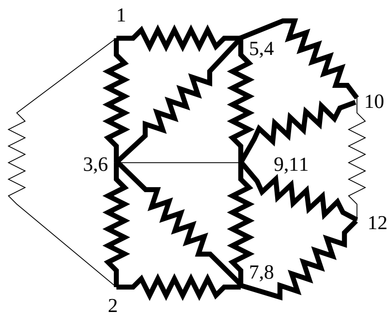
There are two solutions; the second is by far more general and will work with any symmetric polyhedron (or a resistor graph).

(a) A straightforward solution

The trick is to simplify the diagram by connecting with a wire points which have equal potential anyway. Let's assume that the leads are connected to vertices 1 and 2. Then due to symmetry consideration the following sets of points will have equal potentials:

- $3, 6, 9, 11$
- $4, 5$
- $7, 8$

The diagram reduces to:



Thicker pen used for resistances of 1/2 Ohm.
Further simplification is achieved by shorting the middle points of $1-2$ and $10-12$ resistors to the equipotential line $3,6-9,11$. Now the resistance between 1 and 2 can be calculated as (right to left, in Ohm)

$$R_{1-2}/2 = (((1/2||1/2) + 1/2)||1/2||1/2) + 1/2)||1/2||1/2 = 11/60, \qquad (17)$$

where $a||b = ab/(a+b)$
Resistance between the points 1 and 2 is 11/30 Ohm.

(b) A more elegant solution
Consider the following current configuration. Current I is being fed into vortex 1. The other 11 vortices are connected to an external circuit in such a way that each of them drains the same current, *i.e.*, $I/(12-1)$. Symmetry arguments tell us that all of the 5 resistors (edges), emanating at vortex 1 will be carrying the same current, namely $I/5$. The voltage across the resistor $1-2$ will then be $I/5 \times 1Ohm$.
Now let's consider a second configuration. This time, vortex 2 is *drawing* current I, and each of the other vortices $(1, 3-12)$ is connected to an external circuit in such a way that the current flowing into each of them is $I/(12-1)$
The next step is to superimpose the two configurations. It is possible to do because the resistors are linear, *i.e.*, $V = const \times I$. For each resistor, the total current flowing is the sum of the currents in the two configurations. Same holds for the voltage drops. The currents flowing to the outside circuits from vertices $3-12$ are zero, and so the connections to those circuits can be severed.
The current flowing through vertices 1 and 2 are equal in magnitude and are $I + I/(12-1)$. The voltage drop across resistor $1-2$ is $V = 2 \times I/5 \times 1Ohm$.
The resistance measured between points 1 and 2 is then

$$R = \frac{2I/5}{12/11I} Ohm = 11/30 Ohm \qquad (18)$$

Note that the numerator is equal to the total number of vortices minus one, and the denominator is equal to the total number of resistors (edges), the latter being equal to the

$$\frac{1}{2}\, number\ of vortices \times number\ of\ resistors\ eminating\ from\ each\ vortex \tag{19}$$

We leave it to the reader to verify the formula for other symmetric polyhedra.

5. Let the angular speeds of the tops be ω_i, starting with the top one (so $\omega_1 \equiv \omega$). Let I be the moment of inertia of each top around its symmetry axis. Let Ω be the angular speed of precession.

   We will use $\vec{\tau} = d\vec{L}/dt$ on each top. We therefore must determine $d\vec{L}/dt$ and the torque $\vec{\tau}$ for each top.

   - $d\vec{L}/dt$:

     If the ω_i's are large enough (as we are assuming), then the angular momentum of the ith top will have magnitude essentially equal to $L_i = I\omega_i$, and $\vec{L}_i$ will point along the symmetry axis. (In other words, we can neglect the angular momentum due to the slow angular velocity of precession. We will see below that $\Omega \propto 1/\omega$.)

     The tip of $\vec{L}_i$ will trace out a circle of radius $L_i \sin\theta$, with angular speed Ω. Therefore,
$$\left|\frac{d\vec{L}_i}{dt}\right| = \Omega L_i \sin\theta = \Omega I \omega_i \sin\theta, \tag{20}$$
     and $d\vec{L}_i/dt$ points tangentially around the circle.

   - $\vec{\tau}$:

     None of the N tops are accelerating in the vertical direction. Therefore, the forces on the bottom top are NMg upward (to balance the weight of all the tops) at its lower end, and $(N-1)Mg$ downward (to keep up the other $N-1$ tops) at its upper end. The torque on the bottom top (around its CM) therefore has magnitude $(2N-1)Mgr\sin\theta$, where r is half the length of a top. It points perpendicular to the page.

     It is easy to see that the torque on the second-to-bottom top has magnitude $(2N-3)Mgr\sin\theta$, and so on, until the torque on the top top is $Mgr\sin\theta$.

     So the torque on the ith top has magnitude $(2i-1)Mgr\sin\theta$.

   Equating $\vec{\tau}$ with $d\vec{L}_i/dt$ gives
$$(2i-1)Mgr\sin\theta = \Omega I \omega_i \sin\theta. \tag{21}$$
   Therefore,
$$\omega_i = (2i-1)\omega_1 \equiv (2i-1)\omega. \tag{22}$$

   NOTE: As a double-check, the reader can verify that these ω_i's make $\vec{\tau} = d\vec{L}/dt$ true, where $\vec{\tau}$ and $\vec{L}$ are the total torque and angular momentum about the CM of the entire system.

6. (a) Let r be the distance from A to the pulley, and let θ be the angle of the string to A (w.r.t. to the vertical). Let T be the tension in the string (T will depend on r and θ). Then $F = ma$, along the direction of the string, on masses A and B gives, respectively,

$$\begin{aligned} T - mg\cos\theta &= mr\dot{\theta}^2 - m\ddot{r}, \\ T - mg &= m\ddot{r}. \end{aligned} \tag{23}$$

These combine to give $2\ddot{r} = r\dot{\theta}^2 - g(1-\cos\theta)$. Using the small angle approximation for $\cos\theta$, we have

$$2\ddot{r} = r\dot{\theta}^2 - \frac{1}{2}g\theta^2. \tag{24}$$

Consider two cases of the motion.

- *Immediately after A is given the kick:*
  At this time, r is essentially not changing. Hence, the motion is approximately that of a pendulum of length ℓ. Therefore, θ and $\dot{\theta}$ take the form

  $$\theta \approx \frac{\epsilon}{r}\sin\omega t, \qquad \text{and} \qquad \dot{\theta} \approx \frac{\omega\epsilon}{r}\cos\omega t, \tag{25}$$

  where $\omega = \sqrt{g/r}$, and $r = \ell$. Plugging these expressions into eq. (24), and using $\omega^2 = g/r$, gives

  $$2\ddot{r} = \frac{g\epsilon^2}{r^2}\cos^2\omega t - \frac{g\epsilon^2}{2r^2}\sin^2\omega t. \tag{26}$$

  The $\sin^2\omega t$ and $\cos^2\omega t$ terms average to $1/2$ over a few periods. Therefore, the average value of $\ddot{r}$, over a few periods, is

  $$\ddot{r} = \frac{\epsilon^2 g}{8r^2}. \tag{27}$$

  This is positive. Hence, mass B initially starts to climb.
- *After B has risen a significant distance:*
  In this case, r will be changing, so the motion won't look exactly like that of a pendulum.[1] However, it turns out that $\ddot{r}$ is still given by eq. (27), (except that both ϵ and r will have changed; we will show how ϵ changes, in part (b)). Let us see why this is true.

[1] We will find in part (b) that after a while, $\dot{r}$ is roughly the same size as the speed of the pendulum when it passes through $\theta = 0$. So the motion is not like a that of a pendulum.

Eq. (24) is still valid when $\dot{r} \neq 0$. The quantities that require some care are the θ and $\dot{\theta}$ in eq. (25). It is not obvious that these expressions are valid, since the motion doesn't resemble that of a pendulum. However, these forms of θ and $\dot{\theta}$ *are* still true, for the following reason.

At a given time, let $\dot{r} = v$. Consider a frame moving at downward at constant speed v. In this frame, the motion of A looks like that of a pendulum. The acceleration due to gravity in this frame is still g. And most importantly, the fractional change in r, over one period, is very small (because $\dot{r}$ is very small, as we shall see). Hence, the motion looks like that of a pendulum with definite frequency $\omega = \sqrt{g/r}$. And since the frame moves at constant speed, $\ddot{r}$ in this frame equals $\ddot{r}$ is the lab frame. So eq. (27) is still valid. Hence, $\ddot{r}$ is always positive, and so r increases. Therefore, mass B is the one that hits its pulley.

(b) From eq. (27), the initial acceleration of B is $a_i = \epsilon^2 g/8\ell^2$. If this were the acceleration at all times, then the speed of B when it hits the pulley would be $\sqrt{2a_i\ell} = \sqrt{\epsilon^2 g/4\ell}$. This, however, is not correct, because as time goes by both ϵ and ℓ in eq. (27) change, thereby making $\ddot{r}$ change as B rises. To determine how $\ddot{r}$ behaves, it will suffice to determine how ϵ depends on r.

Claim: *The amplitude ϵ scales with r according to $\epsilon \propto r^{1/4}$.*

Proof: We will find ϵ as a function of r by looking at the kinetic energy of A at $\theta = 0$.

The kinetic energy of A, at $\theta = 0$, decreases in time. This is most easily seen by noting that B picks up KE, since it begins to move upward; therefore A must lose KE. (Consider an instant when $\theta = 0$. The total potential energy of the system is the same as when it started. Therefore, the KE gained by B equals the KE lost by A.)

This relationship between the KE's of A and B, at $\theta = 0$, may be expressed as

$$\frac{d}{dr}\left(\frac{1}{2}mr^2\dot{\theta}^2_{\theta=0} + \frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dr}\left(\frac{1}{2}m\dot{r}^2\right). \tag{28}$$

Now, $d\left(\frac{1}{2}m\dot{r}^2\right)$ is the work done on B, which is $dW_B = (T - mg)dr$. From eqs. (23) and (27), we find

$$\frac{dW_B}{dr} = T - mg = \frac{mg\epsilon^2}{8r^2}, \tag{29}$$

where we have taken an average over a few periods to obtain the second equality.

Also, eq. (25) gives $\dot{\theta}^2_{\theta=0} = (\epsilon\omega/r)^2$, with $\omega = \sqrt{g/r}$. So eq. (28) becomes

$$\frac{d}{dr}\left(\frac{mg\epsilon^2}{2r}\right) = -2\frac{dW_B}{dr} = -\frac{mg\epsilon^2}{4r^2}. \tag{30}$$

Taking the derivative and simplifying yields

$$\frac{1}{\epsilon}\frac{d\epsilon}{dr} = \frac{r}{4}. \tag{31}$$

Integrating and then exponentiating gives $\epsilon_r \equiv \epsilon(r) = Cr^{1/4}$. We therefore find that under this very slow change in r, the amplitude ϵ scales like $r^{1/4}$. ■
The initial condition $\epsilon_\ell \equiv \epsilon$, gives

$$\epsilon_r = \epsilon\left(\frac{r}{\ell}\right)^{1/4}. \tag{32}$$

The acceleration in eq. (27) then becomes

$$\ddot{r} = \left(\frac{\epsilon^2 g}{8\sqrt{\ell}}\right)\frac{1}{r^{3/2}}. \tag{33}$$

Multiplying by $\dot{r}$ and integrating gives

$$\frac{\dot{r}^2}{2} = \frac{\epsilon^2 g}{4\ell} - \left(\frac{\epsilon^2 g}{4\sqrt{\ell}}\right)\frac{1}{\sqrt{r}}, \tag{34}$$

where the constant of integration on the right-hand side was determined using the condition that $\dot{r} = 0$ when $r = \ell$.
Finally, plugging in $r = 2\ell$ gives

$$\dot{r}_{r=2\ell} = \sqrt{\frac{\epsilon^2 g}{2\ell}\left(1 - \frac{1}{\sqrt{2}}\right)}. \tag{35}$$

Note: This $\dot{r}$ is the same order of magnitude as the angular speed of the 'pendulum' at $\theta = 0$, namely $r\dot{\theta} = \epsilon_r\omega_r$, because this is (up to factors of order 1) equal to $\epsilon_\ell\omega_\ell = \epsilon\sqrt{g/\ell}$.

The Boston Area Undergraduate Physics Competition

April 17, 1999

Name: ________________________________

School: ________________________________

Year: ________________________________

Address: ________________________________

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Phone: ________________________________

Do not turn this page until you are told to do so.

You have four (4) hours to complete this exam.

Please provide the information requested on this cover sheet. At the end of the exam, hand in this cover sheet with your solutions. You may keep the exam questions.

Show all relevant work in your exam books. Please write neatly. Partial credit will be given for significant progress made toward a correct solution.

You must be enrolled in a full-time undergraduate program to be eligible for prizes.

1999
Boston Area Undergraduate
Physics Competition

April 17, 1999
Time: 4 hours

Each of the six questions is worth 10 points.

1. (a) Consider a solid sphere of uniform charge density. What is the ratio of the electrostatic potential at the surface to that at the center?

   (b) Consider a solid cube of uniform charge density. What is the ratio of the electrostatic potential at a corner to that at the center?

   (Take the potential to be zero at infinity, as usual.)

2. A wheel with spokes rolls on the ground. A stationary camera takes a picture of the wheel. Due to the nonzero exposure time of the camera, the spokes will generally appear blurred. At what location (locations) in the picture does (do) a spoke (the spokes) *not* appear blurred?

3. A motorcyclist wishes to travel in a circle of radius R. The coefficient of static friction between the tires and the (horizontal) ground is constant. The motorcycle starts at rest. What is the minimum distance the motorcycle must travel in order to achieve its maximum allowable speed (that is, the speed above which it must skid out of the circular path)?

4. (a) A fox chases a rabbit. Both run at the same speed v. At all times, the fox runs directly toward the instantaneous position of the rabbit, and the rabbit runs at an angle α relative to the direction directly away from the fox. Their initial separation is ℓ.

   When and where does the fox catch the rabbit (if it does)? If it never does, what is their eventual separation?

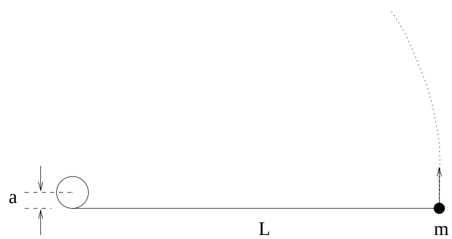
   (b) Consider the same situation, except now let the rabbit always move in the straight line of its initial direction in part (a).

   When and where does the fox catch the rabbit (if it does)? If it never does, what is their eventual separation?

5. Assume that a cloud consists of tiny water droplets suspended in air (uniformly distributed, and at rest), and consider a raindrop falling through them. After a long time, the raindrop moves with constant acceleration. Find this acceleration.

   (Assume that when the raindrop hits a water droplet, the droplet's water gets added to the raindrop. Also, assume that the raindrop is spherical at all times. Ignore air resistance on the raindrop.)

6. A mass m is attached to one end of a spring of zero equilibrium length, the other end of which is fixed. The spring constant is K. Initial conditions are set up so that the mass moves around in a circle of radius L on a frictionless horizontal table. (By “zero equilibrium length”, we mean that the equilibrium length is negligible compared to L.)



At a given time, a vertical pole (of radius a, with $a \ll L$) is fixed onto the table next to the center of the circle, as shown. The spring winds around, and the mass eventually hits the pole. Assume that the pole is sticky, so that any part of the spring touching the pole does not slip. How long does it take for the mass to hit the pole?
(Work in the approximation where $a \ll L$.)

1999
Boston Area Undergraduate
Physics Competition

Solutions

1. (a) Let the sphere have radius R and charge Q. Then the potential at the surface is

$$V(R) = \frac{Q}{R}. \tag{1}$$

The magnitude of the field at radius r inside the sphere is (from Gauss' law)

$$E(r) = \frac{Q(r^3/R^3)}{r^2} = \frac{Qr}{R^3}. \tag{2}$$

Integrating this from $r = R$ down to $r = 0$ gives a change in potential of $\Delta V = Q/2R$. Therefore, the potential at the center is

$$V(0) = \frac{Q}{R} + \frac{Q}{2R} = \frac{3Q}{2R}, \tag{3}$$

and the desired ratio is

$$\frac{V(R)}{V(0)} = \frac{2}{3}. \tag{4}$$

(b) Let ρ be the charge density of the cube. Let V_ℓ^{cor} be the potential at the corner of a cube of side ℓ. Let V_ℓ^{cen} be the potential at the center of a cube of side ℓ. By dimensional analysis,

$$V_\ell^{\text{cor}} \propto \frac{Q}{\ell} = \rho\ell^2. \tag{5}$$

Therefore,[1]

$$V_\ell^{\text{cor}} = 4V_{\ell/2}^{\text{cor}}. \tag{6}$$

But by superposition, we have

$$V_\ell^{\text{cen}} = 8V_{\ell/2}^{\text{cor}}, \tag{7}$$

because the center of the larger cube lies at a corner of the eight smaller cubes of which it is made. Therefore,

$$\frac{V_\ell^{\text{cor}}}{V_\ell^{\text{cen}}} = \frac{4V_{\ell/2}^{\text{cor}}}{8V_{\ell/2}^{\text{cor}}} = \frac{1}{2}. \tag{8}$$

2. The contact point on the ground does not look blurred, because it is instantaneously at rest. However, although this is the only point on the wheel that is at rest, there will be other locations in the picture where the spokes do not appear blurred.

The characteristic of a point in the picture where a spoke does not appear blurred is that the point lies on the spoke during the entire duration of the camera's exposure. (The point

[1] In other words, imagine expanding a cube with side $\ell/2$ to one with side ℓ. If we consider corresponding pieces of the two cubes, then the larger piece has $2^3 = 8$ times the charge of the smaller. But corresponding distances are twice as big in the large cube as in the small cube. Therefore, the larger piece contributes $8/2 = 4$ times as much to V_ℓ^{cor} as the smaller piece contributes to $V_{\ell/2}^{\text{cor}}$.

need not, however, correspond to the same point on the spoke.) At a certain time, consider a spoke in the lower half of the wheel. A short time later, the spoke will have moved, but it will intersect its original position. The spoke will not appear blurred at this intersection point. We must therefore find the locus of these intersections.

Let R be the radius of the wheel. Consider a spoke that makes an angle θ with the vertical. Let the wheel roll through an angle $d\theta$; then the center moves a distance $R\,d\theta$. The spoke's motion is a combination of a translation through a distance $R\,d\theta$, and a rotation through an angle $d\theta$ (about its top end).

Let r be the radial position of the intersection of the initial and final positions of the spoke. Then from the figure we have

$$(R\,d\theta)\cos\theta = r\,d\theta. \tag{9}$$

Therefore, $r = R\cos\theta$. This is easily seen to describe a circle whose diameter is the (lower) vertical radius of the wheel.

3. **First solution (slick method):**

Let β be the angle the force vector makes with the tangential direction. Let F be the maximum possible magnitude of the force of friction (it happens to be $F = \mu mg$, but we won't need this). The minimum-distance scenario is obtained when $F\sin\beta$ accounts for the radial acceleration, and the remaining $F\cos\beta$ accounts for the tangential acceleration. In other words,

$$F\sin\beta = \frac{mv^2}{R}, \qquad \text{and} \qquad F\cos\beta = m\dot{v}. \tag{10}$$

Taking the derivative of the first equation gives $F\cos\beta\dot{\beta} = 2mv\dot{v}/R$. Dividing this by the second equation gives $\dot{\beta} = 2v/R$. But $v = R\dot{\theta}$, where θ is the angular distance traveled around the circle. Therefore, $\dot{\beta} = 2\dot{\theta}$, and integration gives

$$\beta = 2\theta. \tag{11}$$

When the maximum speed is achieved, the value of β must be $\pi/2$. This value corresponds to

$$\theta = \frac{\pi}{4}. \tag{12}$$

Hence, the motorcycle must travel a distance $\pi R/4$, or one-eighth of the way around the circle.

Second solution (straightforward method):

In the minimum-distance scenario, the magnitude of the total force must be its maximum possible value, namely μmg (the exact form of this is not important). Since the radial force is $F_r = mv^2/R$, the tangential $F = ma$ equation is

$$F_t = \sqrt{(\mu mg)^2 - \left(\frac{mv^2}{R}\right)^2} = m\frac{dv}{dt}. \tag{13}$$

Multiplying through by dx, and then rewriting dx/dt as v (where $dx = R\,d\theta$ is the distance along the circle), we obtain[2]

$$dx = \frac{v\,dv}{\sqrt{(\mu g)^2 - \left(\frac{v^2}{R}\right)^2}}. \tag{14}$$

[2] This is simply the work-energy result, because the work is $F_t\,dx$, and the change in kinetic energy is $d(mv^2/2) = mv\,dv$.

Letting $z \equiv v^2/\mu g R$ gives

$$dx = \frac{R\,dz}{2\sqrt{1-z^2}}\,. \tag{15}$$

The maximum allowable speed, V, is obtained from $\mu m g = mV^2/R$. Therefore, $V^2 = \mu g R$, and the corresponding value of z is 1. The desired distance, X, is then

$$\begin{aligned} X = \int_0^X dx &= \int_0^1 \frac{R\,dz}{2\sqrt{1-z^2}} \\ &= \frac{R}{2}\int_0^{\pi/2} d\theta \qquad (\text{letting } z = \sin\theta) \\ &= \frac{\pi R}{4}\,. \end{aligned} \tag{16}$$

4. (a) The relative speed of the fox and rabbit, along the line connecting them, is always $v_{\text{rel}} = v - v\cos\alpha$. Therefore, the total time needed to decrease their separation from ℓ to zero is

$$T = \frac{\ell}{v(1-\cos\alpha)}\,. \tag{17}$$

This is valid unless $\alpha = 0$, in which case the fox never catches the rabbit.

The location of their meeting is a little trickier to obtain. We offer two methods.

First solution (slick method):

Imagine that the rabbit chases another rabbit, which chases another rabbit, etc. Each animal runs at an angle α relative to the direction directly away from the animal chasing it. The initial positions of all the animals lie on a circle, which is easily seen to have radius

$$R = \frac{\ell/2}{\sin(\alpha/2)}\,. \tag{18}$$

The center of the circle is the point, P, which is the vertex of the isosceles triangle with vertex angle α, and with the initial fox and rabbit positions as the other two vertices. By symmetry, the positions of the animals at all times must lie on a circle with center P. Therefore, P is the desired point where they meet. The hypothetical animals simply spiral in to P.

REMARK: An equivalent solution is the following. At all times, the rabbit's velocity vector is obtained by rotating the fox's velocity vector by α. In integrated form, the previous sentence says that the rabbit's net displacement vector is obtained by rotating the fox's net displacement vector by α. The meeting point, P, is therefore the vertex of the above-mentioned isosceles triangle.

Second solution (messier method):

This solution is a little messy, and not too enlightening, so we won't include every detail.

The speed of the rabbit in the direction orthogonal to the line connecting the two animals is $v\sin\alpha$. Therefore, during a time dt, the direction of the fox's motion changes by an angle $d\theta = v\sin\alpha\,dt/\ell_t$, where ℓ_t is the separation at time t. Hence, the change in the fox's velocity has magnitude $|d\mathbf{v}| = v\,d\theta = v(v\sin\alpha\,dt/\ell_t)$. The vector $d\mathbf{v}$ is orthogonal to $\mathbf{v}$; therefore, to get the x-component of $d\mathbf{v}$, we need to multiply

$|d\mathbf{v}|$ by v_y/v. Similar reasoning holds for the y-component of $d\mathbf{v}$, so we arrive at the two equations,

$$\begin{aligned} \dot{v}_x &= \frac{v v_y \sin\alpha}{\ell_t}, \\ \dot{v}_y &= -\frac{v v_x \sin\alpha}{\ell_t}. \end{aligned} \tag{19}$$

Now, we know that $\ell_t = (\ell - v(1-\cos\alpha)t)$. Multiplying the above two equations by ℓ_t, and integrating from the initial to final times (the left sides require integration by parts), yields

$$\begin{aligned} v_{x,0}\ell + v(1-\cos\alpha)X &= v\sin\alpha Y, \\ v_{y,0}\ell + v(1-\cos\alpha)Y &= -v\sin\alpha X, \end{aligned} \tag{20}$$

where (X, Y) is the total displacement vector, and $(v_{x,0}, v_{x,0})$ is the initial velocity vector. Putting all the X and Y terms on the right sides, and squaring and adding the equations, gives

$$\ell^2 v^2 = (X^2 + Y^2)(v^2 \sin^2\alpha + v^2(1-\cos\alpha)^2). \tag{21}$$

Therefore, the net displacement is

$$R = \sqrt{X^2+Y^2} = \frac{\ell}{\sqrt{2(1-\cos\alpha)}} = \frac{\ell/2}{\sin(\alpha/2)}. \tag{22}$$

To find the exact location, we can, with out loss of generality, set $v_{x,0} = 0$, in which case we find $Y/X = (1-\cos\alpha)/\sin\alpha = \tan(\alpha/2)$. This agrees with the result of the first solution.

(b) **First solution (slick method):**

Let $A(t)$ and $B(t)$ be the positions of the fox and rabbit, respectively. Let $C(t)$ be the foot of the perpendicular dropped from A to the line of the rabbit's path. Let α_t be the angle, as a function of time, at which the rabbit moves relative to the direction directly away from the fox (so $\alpha_0 \equiv \alpha$, and $\alpha_\infty = 0$).

The speed at which the distance AB decreases is equal to $v - v\cos\alpha_t$. And the speed at which the distance CB increases is equal to $v - v\cos\alpha_t$. Therefore, the sum of the distances AB and CB does not change. Initially, the sum is $\ell + \ell\cos\alpha$. In the end, it is $2d$, where d is the desired eventual separation. Therefore,

$$d = \frac{\ell(1+\cos\alpha)}{2}. \tag{23}$$

Second solution (straightforward method):

Let α_t be defined as in the first solution, and let ℓ_t be the separation at time t. The speed of the rabbit in the direction orthogonal to the line connecting the two animals is $v\sin\alpha_t$. The separation is ℓ_t, so the angle α_t changes at a rate

$$\dot{\alpha}_t = -\frac{v\sin\alpha_t}{\ell_t}. \tag{24}$$

And ℓ_t changes at a rate

$$\dot{\ell}_t = -v(1-\cos\alpha_t). \tag{25}$$

Taking the quotient of the above two equations, and separating variables, gives

$$\frac{\dot{\ell}_t}{\ell_t} = \frac{\dot{\alpha}_t(1-\cos\alpha_t)}{\sin\alpha_t}. \tag{26}$$

The right side may be rewritten as $\dot{\alpha}_t \sin\alpha_t/(1+\cos\alpha_t)$, and so integration gives

$$\ln(\ell_t) = -\ln(1+\cos\alpha_t) + C, \tag{27}$$

where C is a constant of integration. Exponentiating gives $\ell_t(1+\cos\alpha_t) = B$. The initial conditions demand that $B = \ell_0(1+\cos\alpha_0) \equiv \ell(1+\cos\alpha)$. Therefore,

$$\ell_t = \frac{\ell(1+\cos\alpha)}{(1+\cos\alpha_t)}. \tag{28}$$

Setting $t=\infty$, and using $\alpha_\infty = 0$, gives the final result

$$\ell_\infty = \frac{\ell(1+\cos\alpha)}{2}. \tag{29}$$

REMARK: The solution of part b) is valid for all α except $\alpha{=}\pi$. If $\alpha{=}\pi$, the rabbit runs directly towards fox and they will indeed meet halfway in time $\ell/2v$.

5. Let ρ be the mass density of the raindrop, and let λ be the average mass density in space of the water droplets. Let $r(t)$, $M(t)$, and $v(t)$ be the radius, mass, and speed of the raindrop, respectively.

   The mass of the raindrop is $M = (4/3)\pi r^3\rho$. Therefore,

$$\dot{M} = 4\pi r^2\dot{r}\rho = 3M\frac{\dot{r}}{r}. \tag{30}$$

   Another expression for $\dot{M}$ is obtained by noting that the change in M is due to the acquisition of water droplets. The raindrop sweeps out volume at a rate given by its cross-sectional area times its velocity. Therefore,

$$\dot{M} = \pi r^2 v\lambda. \tag{31}$$

   The force of Mg on the droplet equals the rate of change of its momentum, namely $dp/dt = d(Mv)/dt = \dot{M}v + M\dot{v}$. Therefore,

$$Mg = \dot{M}v + M\dot{v}. \tag{32}$$

   We now have three equations involving the three unknowns, r, M, and v.

   (**Note:** We *cannot* write down the naive conservation-of-energy equation, because mechanical energy is *not* conserved. The collisions between the raindrop and the droplets are completely inelastic. The raindrop will, in fact, heat up. See the remark at the end of the solution.)

   The goal is to find $\dot{v}$ for large t. We will do this by first finding $\ddot{r}$ at large t. Eqs. (30) and (31) give

$$v = \frac{4\rho}{\lambda}\dot{r} \qquad \Longrightarrow \qquad \dot{v} = \frac{4\rho}{\lambda}\ddot{r}. \tag{33}$$

Plugging eqs. (30) and (33) into eq. (32) gives

$$Mg = \left(3M\frac{\dot{r}}{r}\right)\left(\frac{4\rho}{\lambda}\dot{r}\right) + M\left(\frac{4\rho}{\lambda}\ddot{r}\right). \tag{34}$$

Therefore,

$$\frac{g\lambda}{\rho}r = 12\dot{r}^2 + 4r\ddot{r}. \tag{35}$$

Given that the raindrop falls with constant acceleration at large times, we may write [3]

$$\ddot{r} \approx bg, \qquad \dot{r} \approx bgt, \qquad \text{and} \qquad r \approx \frac{1}{2}bgt^2, \tag{36}$$

for large t, where b is a numerical factor to be determined. Plugging eqs. (36) into eq. (35) gives

$$\left(\frac{g\lambda}{\rho}\right)\left(\frac{1}{2}bgt^2\right) = 12(bgt)^2 + 4\left(\frac{1}{2}bgt^2\right)bg. \tag{37}$$

Therefore, $b = \lambda/28\rho$. Hence, $\ddot{r} = g\lambda/28\rho$, and eq. (33) gives the acceleration of the raindrop at large t,

$$\dot{v} = \frac{g}{7}, \tag{38}$$

independent of ρ and λ.

Remark: We can calculate how much mechanical energy is lost (and therefore how much the raindrop heats up) as a function of the height fallen.

The fact that v is proportional to $\dot{r}$ (shown in eq. (33)) means that the volume swept out by the raindrop is a cone. The center-of-mass of a cone is 1/4 of the way from the base to the apex. Therefore, if M is the mass of the raindrop after it has fallen a height h, then the loss in mechanical energy is

$$E_{\text{lost}} = Mg\frac{h}{4} - \frac{1}{2}Mv^2. \tag{39}$$

Using $v^2 = 2(g/7)h$, this becomes

$$\Delta E_{\text{int}} = E_{\text{lost}} = \frac{3}{28}Mgh, \tag{40}$$

where ΔE_{int} is the gain in internal thermal energy. The energy required to heat 1g of water by 1 degree is 1 calorie (= 4.18 Joules). Therefore, the energy required to heat 1 kg of water by 1 degree is ≈ 4200 J. In other words,

$$\Delta E_{\text{int}} = 4200\, M\, \Delta T, \tag{41}$$

where mks units are used, and T is measured in celsius. (We have assumed that the internal energy is uniformly distributed throughout the raindrop.) Eqs. (40) and (41) give the increase in temperature as a function of h,

$$4200\,\Delta T = \frac{3}{28}gh. \tag{42}$$

How far must the raindrop fall before it starts to boil? If we assume that the water droplets' temperature is near freezing, then the height through which the raindrop must fall to have $\Delta T = 100\,^\circ\text{C}$ is found to be

$$h = 400\,\text{km}. \tag{43}$$

[3] We may justify the constant-acceleration statement in the following way. For large t, let r be proportional to t^α. Then the left side of eq. (35) goes like t^α, while the right side goes like $t^{2\alpha-2}$. If these are to be equal, then we must have $\alpha = 2$. Hence, $r \propto t^2$, and $\ddot{r}$ is a constant (for large t).

We have, of course, idealized the problem. But needless to say, there is no need to worry about getting burned by the rain.

A typical value for h is 10 km, which would raise the temperature by two or three degrees. This effect, of course, is washed out by many other factors.

6. Let $\theta(t)$ be the angle through which the spring moves. Let $x(t)$ be the length of the unwrapped part of the spring. Let $v(t)$ be the speed of the mass. And let $k(t)$ be the spring constant of the unwrapped part of the spring. (The manner in which k changes will be derived below.)

Using the approximation $a \ll L$, we may say that the mass undergoes approximate circular motion. (This approximation will break down when x becomes of order a, but the time during which this is true is negligible compared to the total time.) The instantaneous center of the circle is the point where the spring touches the pole. $F = ma$ along the instantaneous radial direction gives

$$\frac{mv^2}{x} = kx. \tag{44}$$

Using this value of v, the frequency of the circular motion is given by

$$\omega \equiv \frac{d\theta}{dt} = \frac{v}{x} = \sqrt{\frac{k}{m}}. \tag{45}$$

The spring constant, $k(t)$, of the unwrapped part of the spring is inversely proportional to its equilibrium length. (For example, if you cut a spring in half, the resulting springs have twice the original spring constant). All equilibrium lengths in this problem are infinitesimally small (compared to L), but the inverse relation between k and equilibrium length still holds. (If you want, you can think of the equilibrium length as a measure of the total number of spring atoms that remain in the unwrapped part.)

Note that the change in angle of the contact point on the pole equals the change in angle of the mass around the pole (which is θ.) Consider a small interval of time during which the unwrapped part of the spring stretches a small amount and moves through an angle $d\theta$. Then a length $a\,d\theta$ becomes wrapped on the pole. So the fractional decrease in the equilibrium length of the unwrapped part is (to first order in $d\theta$) equal to $(a\,d\theta)/x$. From the above paragraph, the new spring constant is therefore

$$k_{\text{new}} = \frac{k_{\text{old}}}{1 - \frac{a\,d\theta}{x}} \approx k_{\text{old}}\left(1 + \frac{a\,d\theta}{x}\right). \tag{46}$$

Therefore, $dk = ka\,d\theta/x$. Dividing by dt gives

$$\dot{k} = \frac{ka\omega}{x}. \tag{47}$$

The final equation we need is the one for energy conservation. At a given instant, consider the sum of the kinetic energy of the mass, and the potential energy of the unwrapped part of the spring. At a time dt later, a tiny bit of this energy will be stored in the newly-wrapped little piece. Letting primes denote quantities at this later time, conservation of energy gives

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}k'x'^2 + \frac{1}{2}m'v'^2 + \left(\frac{1}{2}kx^2\right)\left(\frac{a\,d\theta}{x}\right). \tag{48}$$

The last term is (to lowest order in $d\theta$) the energy stored in the newly-wrapped part, because $a\,d\theta$ is its length. Using eq. (44) to write the v's in terms of the x's, this becomes

$$kx^2 = k'x'^2 + \frac{1}{2}kxa\,d\theta. \tag{49}$$

In other words, $-(1/2)kxa\,d\theta = d(kx^2)$. Dividing by dt gives

$$\begin{aligned} -\frac{1}{2}kxa\omega &= \frac{d(kx^2)}{dt} \\ &= \dot{k}x^2 + 2kx\dot{x} \\ &= \left(\frac{ka\omega}{x}\right)x^2 + 2kx\dot{x}, \end{aligned} \tag{50}$$

where we have used eq. (47). Therefore,

$$\dot{x} = -\frac{3}{4}a\omega. \tag{51}$$

We must now solve the two couple differential equations, eqs. (47) and (51). Dividing the latter by the former gives

$$\frac{\dot{x}}{x} = -\frac{3}{4}\frac{\dot{k}}{k}. \tag{52}$$

Integrating and exponentiating gives

$$k = \frac{KL^{4/3}}{x^{4/3}}, \tag{53}$$

where the numerator is obtained from the initial conditions, $k = K$ and $x = L$. Plugging eq. (53) into eq. (51), and using $\omega = \sqrt{k/m}$, gives

$$x^{2/3}\dot{x} = -\frac{3aK^{1/2}L^{2/3}}{4m^{1/2}}. \tag{54}$$

Integrating, and using the initial condition $x = L$, gives

$$x^{5/3} = L^{5/3} - \left(\frac{5aK^{1/2}L^{2/3}}{4m^{1/2}}\right)t. \tag{55}$$

So, finally,

$$x(t) = L\left(1 - \frac{t}{T}\right)^{3/5}, \tag{56}$$

where

$$T = \frac{4}{5}\frac{L}{a}\sqrt{\frac{m}{K}} \tag{57}$$

is the time for which $x(t) = 0$ and the mass hits the pole.

REMARKS:

(a) Note that the angular momentum of the mass around the center of the pole is *not* conserved in this problem, because the force is not a central force.

(b) Integrating eq. (51) up to the point when the mass hit the pole gives $-L = -(3/4)a\theta$. But $a\theta$ is the total length wrapped around the pole, which we see is equal to $4L/3$.

Boston Area Undergraduate
Physics Competition

April 22, 2000
Time: 4 hours

Each of the six questions is worth 10 points.

1. A thin stick with length L (and uniform mass density) is pivoted at a point P. The stick is held horizontal and then released. Where should P be located so that the stick swings down and passes through the vertical position in the minimum time?

2. (a) Two conducting infinite half-planes meet at a right angle. A charge q is brought in from rest at infinity, to a position (at rest) a distance d from each plane. What is the work done, W_{in}, to bring about this change?

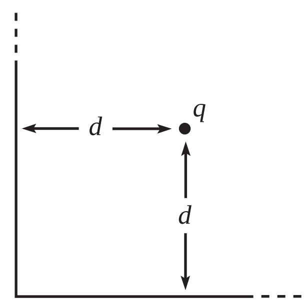


Figure 1: Problem 2

   (b) While the charge q is a distance d from each plane, the planes are changed from conducting to insulating (that is, the charges on them are no longer free to move). The charge q is then brought back out to infinity. What is the work done, W_{out}, to bring about this change?

   (c) What is the potential energy of the system of charges on the insulating planes?

3. Consider a planet which is made of matter with the same average density as the earth. Assume that the atmospheric pressure on the planet's surface is the same as on the earth's surface. Assume for simplicity that the planet's atmospheric temperature is independent of height, and that it is equal to the temperature at the surface of the earth. And assume that the composition of the atmosphere on the planet is the same as on the earth.

   What should the radius of the planet be, so that a light beam can travel in a circle around the planet, just above its surface?

   Note: You will need to use the fact that the index of refraction depends on the density of air according to $n(\rho) = 1 + \epsilon\rho$, where ϵ is a given constant.

   Give your answer in terms of:

R_E, the radius of the earth,

g_E, the acceleration at the surface of the earth,

P_E, the atmospheric pressure at the surface of the earth,

ρ_E, the atmospheric density at the surface of the earth, and

ϵ.

4. N identical balls lie equally spaced in a semicircle, on a frictionless horizontal table, as shown. The total mass of these balls is M. Another ball of mass m approaches the semicircle from the left, with the proper initial conditions so that it bounces (elastically) off all N balls and finally leaves the semicircle, heading directly to the left.

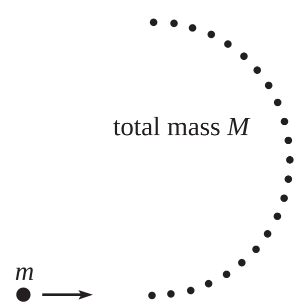


Figure 2: Problem 4

   (a) In the limit $N \to \infty$ (so the mass of each ball in the semicircle, M/N, goes to zero), find the minimum value of M/m which allows the incoming ball to come out heading directly to the left.

   (b) In the limiting case found in part (a), find the ratio of m's final speed to initial speed.

5. A turntable rotates with constant angular speed Ω. A ball rolls on it, without slipping. The ball has uniform mass density, so that its moment of inertia is $I = (2/5)MR^2$.

   Show that whatever the (non-slipping) initial conditions are, the ball will move in a circle (as viewed from the inertial lab frame). What is the frequency of this circular motion?

6. A rope of mass density σ hangs from a spring with spring-constant k. In the equilibrium position, the bottom part of the rope lies in a heap on the floor, and a length L is in the air. The top of the spring is held fixed.

   The rope is raised by a very small distance b and then released. What is the amplitude of oscillations, as a function of time?

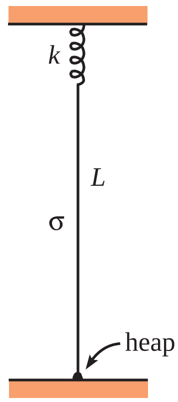


Figure 3: Problem 6

(Assume the following: (1) $L \gg b$, (2) the rope is very thin, so that the size of the heap on the floor is very small compared to b, (3) the length of the rope in the initial heap is larger than b, so that some of the rope always remains in contact with the floor, and (4) there is no friction of the rope with itself inside the heap.)

Boston Area Undergraduate
Physics Competition
April 22, 2000

SOLUTIONS

1. Let x be the distance from P to the center of the stick. The moment of inertia of the stick (around the center of mass) is

$$I_{\rm CM} = \int_{-L/2}^{L/2} x^2\,dm = \int_{-L/2}^{L/2} x^2\sigma\,dx = \frac{(\sigma L)L^2}{12} = \frac{ML^2}{12}\,. \tag{1}$$

Using the parallel-axis theorem, the moment of inertia around point P is

$$I_P = \frac{ML^2}{12} + Mx^2. \tag{2}$$

The torque, relative to P, is due to gravity effectively acting at the center of mass. Therefore, when the stick makes an angle θ with respect to the horizontal, the torque is $\tau = Mgx\cos\theta$. Hence, $\tau = I_P\alpha$ gives

$$\alpha = \frac{Mgx\cos\theta}{\frac{ML^2}{12} + Mx^2}\,. \tag{3}$$

The stick will fall quickest when the coefficient of $\cos\theta$ is maximum. Taking the derivative with respect to x, we find that α is maximized when

$$x = \frac{L}{\sqrt{12}}\,. \tag{4}$$

2. (a) Add three image charges to create the square of charges shown below. It is easy to see that the total electric field caused by all four charges is perpendicular to the two planes. Since this field satisfies the boundary conditions required by the conducting planes, it must be the same (due to the uniqueness theorem) as the field produced by the actual charge q and the negative charges that build up on the planes.

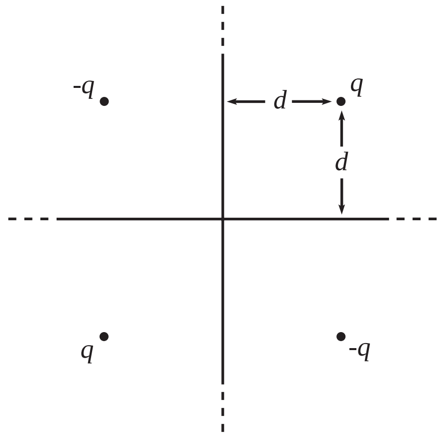
The total potential energy of the entire system of charges is

$$V_{\rm tot} = 4\left(\frac{-q^2}{2d}\right) + 2\left(\frac{q^2}{2\sqrt{2}\,d}\right). \tag{5}$$

The potential energy of the actual charge q and the negative charges that build up on the planes is $V_{\rm tot}/4$. (This can be seen by noting that the energy of a system of charges is equal to the integral of the square of the electric field. And the actual setup has an electric field in only one quadrant of space.) The work $W_{\rm in}$ is equal to the potential energy of the actual system of charges. Therefore,

$$W_{\rm in} = \frac{V_{\rm tot}}{4} = \frac{q^2}{d}\left(-\frac{1}{2} + \frac{1}{4\sqrt{2}}\right), \tag{6}$$

which is negative, as it should be.



(b) When the charge q is moved away from the insulating planes, it feels a force as if it is interacting with the three image charges fixed at their positions. (This is true because at any point in the upper right quadrant, the electric field due to the image charges must be precisely equal to the electric field due to the charges on the insulating plates. This is why we picked these image charges, after all.)

The work $W_{\rm out}$ is thus equal to the work done in separating the given charge from the three image charges. This work is

$$\begin{aligned} W_{\rm out} &= 2\left(\frac{q^2}{2d}\right) - \left(\frac{q^2}{2\sqrt{2}\,d}\right) \\ &= \frac{q^2}{d}\left(1 - \frac{1}{2\sqrt{2}}\right), \end{aligned} \tag{7}$$

which is positive, as it should be.

(c) The potential energy of the system of charges on the insulating planes is equal to the total work done on the system, which is

$$W_{\rm in} + W_{\rm out} = \frac{q^2}{d}\left(\frac{1}{2} - \frac{1}{4\sqrt{2}}\right), \tag{8}$$

which is positive, as it should be.

3. The condition that light is able to take a circular path around the planet is that this circular path takes the least amount of time, compared to all nearby paths. (This is Fermat's principle of least time.) This condition then implies that nearby circular paths take the same amount of time (to first order in their size difference, at least).

The speed of light in a medium is proportional to the reciprocal of the index of refraction. Therefore, the condition for the existence of a circular path is[1]

$$\frac{n(R+h)}{n(R)} = \frac{R}{R+h}\,. \tag{9}$$

Expanding this to first order in h gives $R = -n/(dn/dh)$. But the given information $n = 1 + \epsilon\rho$ says that $dn/dh = \epsilon\, d\rho/dh$. Therefore,

$$R = -\frac{n}{\epsilon\, d\rho/dh}\,. \tag{10}$$

We must therefore find $d\rho/dh$. We will need three facts.

(a) The first is

$$\frac{d\rho}{dh} = \frac{\rho_E}{P_E}\frac{dP}{dh}\,. \tag{11}$$

This follows from the ideal gas law, $PV = nRT$. Dividing through by V, we find $P \propto \rho$. Since the temperature of the planet's atmosphere is independent of height, the constant of proportionality is independent of height. And since the temperature is the same as on the earth's surface, we have $\rho/P = \rho_E/P_E$, from which eq. (11) follows.

(b) The second is

$$\frac{dP}{dh} = -g\rho. \tag{12}$$

This follows from the usual consideration of a small column of air (with height dh and base area A), which has a net force on it equal to $-A(dP/dh)\,dh - (\rho A\,dh)g$. Setting this equal to zero yields eq. (12).

(c) The third is

$$g = \frac{g_E}{R_E}R. \tag{13}$$

This follows from the definition $g \equiv GM/R^2$, along with the assumption that the densities of the earth and the planet are equal. Writing M as the density times $4\pi R^3/3$ gives $g \propto R$, from which eq. (13) follows.

These three facts imply

$$\frac{d\rho}{dh} = -\left(\frac{\rho_E}{P_E}\right)\left(\frac{g_E}{R_E}R\right)\rho. \tag{14}$$

Plugging this into eq. (10) gives

$$R = \frac{nR_E P_E}{\epsilon g_E \rho_E \rho R}\,. \tag{15}$$

[1] You can also derive this result by considering the atmosphere to consist of layers with different n's, and demanding that there be total internal reflection in a given layer.

Using $n = 1 + \epsilon\rho$, and also the assumption $\rho = \rho_E$, and then solving for R gives

$$R = \sqrt{\frac{(1+\epsilon\rho_E)R_E P_E}{\epsilon g_E \rho_E^2}}. \tag{16}$$

REMARK: The quantity $\epsilon\rho_E$ is roughly equal to $3 \cdot 10^{-4}$. Using this value (its effect in the numerator of eq. (16) is negligible), along with $R_E \approx 6.4 \cdot 10^6\,\text{m}$, $g_E \approx 10\,\text{m/s}^2$, $P_E \approx 1 \cdot 10^5\,\text{kg/ms}^2$, and $\rho_E \approx 1.2\,\text{kg/m}^3$, gives

$$R \approx 1.3 \cdot 10^7\,\text{m} \approx 2R_E. \quad \blacksquare \tag{17}$$

4. (a) Let $\mu \equiv M/N$ be the mass of each ball in the semicircle. We want the deflection angle in each collision to be $\theta = \pi/N$. However, if the ratio μ/m is too small, then this angle of deflection is not possible. Let's be quantitative about this.

Lemma: If a mass m collides with a stationary mass μ, then the maximum angle of deflection is given by

$$\sin\theta_{\text{max}} = \frac{\mu}{m}. \tag{18}$$

Proof: Let V be the initial speed of mass m in the lab frame. Then in the CM frame, it is easy to see that m and μ head toward each other with initial speeds

$$v_m = \frac{\mu V}{m+\mu}, \qquad \text{and} \qquad v_\mu = \frac{mV}{m+\mu}. \tag{19}$$

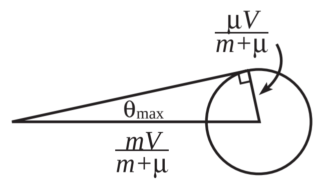
Because the collision is elastic, these speeds are also the final speeds in the CM frame, independent of the final directions of motion (which are of course opposites of each other.)

To return to the lab frame, we must add on the sideways speed of $v_\mu = mV/(m+\mu)$ to each particle. Therefore, the final velocity of m in the lab frame is the vector sum,

$$\mathbf{V}_f = \left(\frac{mV}{m+\mu}\right)\hat{\mathbf{x}} + \left(\frac{\mu V}{m+\mu}\right)\hat{\mathbf{r}}, \tag{20}$$

where $\hat{\mathbf{r}}$ is the unit vector representing the direction on the CM frame. To maximize the angle of deflection in the lab frame, we clearly want to have the situation show below, where the total velocity vector is tangent to the circle. In this case, we have $\sin\theta_{\text{max}} = \mu/m$, as was to be shown.[2]

[2]You can also derive this result by going through the straightforward (but tedious) calculation using conservation of energy and momentum during the collision. You will find that a solution exists (for, say, the final speed of m) only if $\sin\theta \le \mu/m$.



(This result holds for $\mu < m$. If $\mu > m$, then the maximum angle of deflection is $180°$.) ■

In the problem at hand, we have $\theta = \pi/N$. Since θ is small, we may use $\sin\theta \approx \theta$ to write the $\sin\theta \le \mu/m$ condition as

$$\theta \le \frac{\mu}{m} \qquad \Longrightarrow \qquad \frac{\pi}{N} \le \frac{(M/N)}{m} \qquad \Longrightarrow \qquad \pi \le \frac{M}{m}. \tag{21}$$

(Given m, it is clear that there must be a lower bound on M, because if M is very small, the mass m will simply plow through the semicircle.)

(b) From the above figure, the final speed after one bounce, in the case of maximum deflection angle, is

$$V_f = V\frac{\sqrt{m^2 - \mu^2}}{m + \mu} \approx V\left(1 - \frac{\mu}{m}\right), \tag{22}$$

to first order in the small quantity μ/m. The same reasoning holds for all bounces, so the speed decreases by the factor $(1 - \mu/m)$ after each bounce. In the special case where $\mu/m = \pi/N$, we see that after the N bounces off all the balls, the ratio of m's final speed to initial speed is

$$\frac{V_{\text{final}}}{V_{\text{initial}}} = \left(1 - \frac{\pi}{N}\right)^N \approx e^{-\pi}. \tag{23}$$

It doesn't get any nicer than that!
($e^{-\pi}$ is roughly equal to 1/23, so only about 4% of the initial speed remains.)

5. The angular velocity of the turntable is $\Omega\hat{\mathbf{z}}$. Let the angular velocity of the ball be $\boldsymbol{\omega}$. If the ball is at position $\mathbf{r}$ (with respect to the lab frame), then its velocity (with respect to the lab frame) may be broken up into the velocity of the turntable (at position $\mathbf{r}$) plus the ball's velocity relative to the turntable. The non-slipping condition says that this latter velocity is given by $\boldsymbol{\omega} \times (a\hat{\mathbf{z}})$. (We'll use "$a$" to denote the radius of the sphere.) The ball's velocity with respect to the lab frame is thus

$$\mathbf{v} = (\Omega\hat{\mathbf{z}}) \times \mathbf{r} + \boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \tag{24}$$

The angular momentum of the ball is

$$\mathbf{L} = I\boldsymbol{\omega}. \tag{25}$$

The friction force from the ground is responsible for changing both the momentum and the angular momentum of the ball. $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}\,, \tag{26}$$

and $\boldsymbol{\tau} = d\mathbf{L}/dt$ (relative to the center of the ball) gives

$$(-a\hat{\mathbf{z}}) \times \mathbf{F} = \frac{d\mathbf{L}}{dt}\,, \tag{27}$$

since the force is applied at position $-a\hat{\mathbf{z}}$ relative to the ball's center.

We will now use the previous four equations to demonstrate that the ball undergoes circular motion. Our goal will be to produce an equation of the form $d\mathbf{v}/dt = \Omega'\hat{\mathbf{z}} \times \mathbf{v}$, since this describes circular motion, with frequency Ω' (to be determined).

Plugging the expressions for $\mathbf{L}$ and $\mathbf{F}$ from eqs. (25) and (26) into eq. (27) gives

$$\begin{aligned}(-a\hat{\mathbf{z}}) \times \left(m\frac{d\mathbf{v}}{dt}\right) &= I\frac{d\boldsymbol{\omega}}{dt} \\ \Longrightarrow \qquad \frac{d\boldsymbol{\omega}}{dt} &= -\left(\frac{am}{I}\right)\hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt}\,.\end{aligned} \tag{28}$$

Taking the derivative of eq. (24) then gives

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times (a\hat{\mathbf{z}}) \\ &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\left(\frac{am}{I}\right)\hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt}\right) \times (a\hat{\mathbf{z}}).\end{aligned} \tag{29}$$

Since the vector $d\mathbf{v}/dt$ lies in the horizontal plane, it is easy to work out the cross-product in the right term (or just use the identity $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$) to obtain

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\frac{a^2m}{I}\right)\frac{d\mathbf{v}}{dt} \\ \Longrightarrow \qquad \frac{d\mathbf{v}}{dt} &= \left(\frac{\Omega}{1 + (a^2m/I)}\right)\hat{\mathbf{z}} \times \mathbf{v}.\end{aligned} \tag{30}$$

For a uniform sphere, $I = (2/5)ma^2$, so we obtain

$$\frac{d\mathbf{v}}{dt} = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}} \times \mathbf{v}. \tag{31}$$

The ball therefore undergoes circular motion, with a frequency equal to $2/7$ times the frequency of the turntable. This result for the frequency does not depend on initial conditions.

Remarks: Integrating eq. (31) from the initial time to some later time gives

$$\mathbf{v} - \mathbf{v}_0 = \left(\frac{2}{7}\Omega\right) \hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}_0). \tag{32}$$

This may be written in the more suggestive form,

$$\mathbf{v} = \left(\frac{2}{7}\Omega\right) \hat{\mathbf{z}} \times \left(\mathbf{r} - \left(\mathbf{r}_0 + \frac{7}{2\Omega}(\hat{\mathbf{z}} \times \mathbf{v}_0)\right)\right). \tag{33}$$

This equation describes circular motion, with the center located at the point

$$\mathbf{r}_c = \mathbf{r}_0 + (7/2\Omega)(\hat{\mathbf{z}} \times \mathbf{v}_0), \tag{34}$$

and with radius $(7/2\Omega)|\hat{\mathbf{z}} \times \mathbf{v}_0| = 7v_0/2\Omega$. (Eq. (33) does indeed describe circular motion, because it says that $\mathbf{v}$ is always perpendicular to $\mathbf{r} - \mathbf{r}_c$. Hence, the distance to the point $\mathbf{r}_c$ doesn't change.)

There are a few special cases to consider:

- If $v_0 = 0$ (that is, if the spinning motion of the ball exactly cancels the rotational motion of the turntable), then the ball will always remain in the same place (of course).
- If the ball is initially not spinning, and just moving along with the turntable, then $v_0 = \Omega r_0$, so the radius of the circle is $(7/2)r_0$.
- If we want the center of the circle be the center of the turntable, then eq. (34) say that we need $(7/2\Omega)\hat{\mathbf{z}} \times \mathbf{v}_0 = -\mathbf{r}_0$. This implies that $\mathbf{v}_0$ has magnitude $v_0 = (2/7)\Omega r_0$ and points tangentially in the same direction as the turntable moves. (That is, the ball moves at 2/7 times the velocity of the turntable beneath it.)

The fact that the frequency $(2/7)\Omega$ is a rational multiple of Ω means that the ball will eventually return to the same point on the turntable. In the lab frame, the ball will trace out two circles in the time it takes the turntable to undergo seven revolutions. And from the point of view of someone on the turntable, the ball will 'spiral' around five times before returning to the original position.

If we look at a ball with moment of inertia $I = \eta m a^2$ (so a uniform sphere has $\eta = 2/5$), then it is easy to show that the "2/7" in the above result gets replaced by "$\eta/(1+\eta)$". If a ball has most of its mass concentrated at its center (so that $\eta \to 0$), then the frequency of the circular motion goes to 0, and the radius goes to ∞. ■

6. We will divide the solution into the calculations of (1) the frequency of oscillation, (2) the energy loss per oscillation, (3) the amplitude as a function of time.

Frequency of oscillation

In the equilibrium position, the upward force from the spring balances the downward force from gravity on the part of the rope that is in the air. If the rope is moved a distance y (with upward taken to be positive), then the force from the spring changes by $-ky$, while the gravitational force changes by $-(\sigma y)g$. The net force is therefore $F = -(k + \sigma g)y$. This force acts on the rope, which has mass $M = \sigma L$. $F = ma$ therefore gives (the mass of the rope in the air changes slightly, but this effect is negligible when dealing with the "m" in $F = ma$)

$$-(k + \sigma g)y = (\sigma L)\ddot{y}, \tag{35}$$

and so the frequency of oscillation is

$$\omega = \sqrt{\frac{k + \sigma g}{\sigma L}} = \sqrt{\frac{k}{M} + \frac{g}{L}}. \tag{36}$$

REMARKS: A common incorrect answer for the frequency is $\omega = \sqrt{k/M}$. The g/L term definitely belongs in the correct answer, as can be seen by considering the limit $k \to 0$. (That is, we have a very weak spring which is stretched, say, a kilometer. And a rope of, say, 1 meter hangs from the end.) The spring force doesn't vary much with distance, so it will always pull up with a force of essentially $Mg = (L\sigma)g$. If the rope is moved distance y, then the gravitational force equals $-(L + y)\sigma g$. The net force is therefore $-(\sigma g)y$. $F = ma$ then gives $-(\sigma g)y = (L\sigma)\ddot{y}$. Hence $\omega = \sqrt{g/L}$, which is independent of k (even though the spring force is *not* negligible). The rope will simply bounce up and down, with a frequency determined by its length.

We can be a bit more rigorous in deriving the frequency in eq. (36), by writing down the precise equations of motion for the moving part of the rope. If we let $\ell \equiv L + y$ be the length of rope in the air, then $F = dp/dt$ gives

$$F_{\text{net}} = \frac{d}{dt}\Big((\ell\sigma)\dot{\ell}\Big) = \sigma\ell\ddot{\ell} + \sigma\dot{\ell}^2 = \sigma(L + y)\ddot{y} + \sigma\dot{y}^2 \tag{37}$$

On the way up, the net force on the moving part of the rope is $F_{\text{net}} = -(k + \sigma g)y$, so eq. (37) gives

$$-(k + \sigma g)y = \sigma(L + y)\ddot{y} + \sigma\dot{y}^2. \tag{38}$$

On the way down, the net force on the moving part of the rope is $F_{\text{net}} = -(k + \sigma g)y + F_{\text{floor}}$, where F_{floor} is the force exerted by the floor to bring to rest the atoms that hit the floor. Mass hits the floor at a rate $\sigma|\dot{y}|$, while moving at speed $|\dot{y}|$. The rate of change of momentum of these atoms (i.e., F_{floor}) is therefore $\sigma\dot{y}^2$. Eq. (37) then gives

$$-(k + \sigma g)y = \sigma(L + y)\ddot{y}. \tag{39}$$

To first order in the small quantity y, both eqs. (38) and (39) give the equation of motion in eq. (35), which was derived using the approximate $F = ma$ reasoning, with $m = \sigma L$. ■

Energy loss per oscillation

The position of the rope, relative to the equilibrium position, is essentially equal to

$$x(t) = A(t)\cos(\omega t). \tag{40}$$

The energy loss during the downward motion is fairly straightforward. When a piece of the rope with mass dm hits the floor, it loses a kinetic energy of $(1/2)(dm)v^2$. In a short time dt, we have $dm = |\sigma v\, dt|$. So the loss is $|(1/2)\sigma v^3\, dt|$. From eq. (40), we obtain $v(t) = -\omega A(t)\sin(\omega t)$, so the change in energy during the downward half of the oscillation is

$$\Delta E_{\text{down}} = -\frac{1}{2}\int_0^{\pi/\omega} \sigma\omega^3 A^3 \sin^3(\omega t)\, dt. \tag{41}$$

Letting $\theta \equiv \omega t$, and then using

$$\int_0^{\pi} \sin^3\theta\, d\theta = \int_0^{\pi}(1-\cos^2\theta)\sin\theta\, d\theta = \left(-\cos\theta + \frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} = \frac{4}{3}, \tag{42}$$

gives (using the fact that A is essentially constant throughout an oscillation)

$$\Delta E_{\text{down}} = -\frac{2}{3}\sigma\omega^2 A^3. \tag{43}$$

The energy loss during the upward motion is a little trickier, but the answer turns out to be the same as for the downward motion. When a piece of the rope with mass dm abruptly joins the moving part of the rope, there is an inevitable energy loss. This loss may be calculated as follows. Let a mass dm join the rope at the instant the rope is moving at speed v. Then it gains a kinetic energy of $(1/2)(dm)v^2$. It also gains a momentum of $dP = (dm)v$. The work the spring does in bringing it up to this speed is $W = \int F\, dx = \int Fv\, dt$. The rope is moving at an essentially constant speed v for this short period of time. Hence,

$$W = v\int F\, dt = v(dP) = (dm)v^2. \tag{44}$$

We therefore conclude that half of this work goes into kinetic energy of the mass, and half is lost to heat. The loss to heat is thus $(1/2)(dm)v^2 = (1/2)|\sigma v\, dt|v^2 = |(1/2)\sigma v^3\, dt|$, as in the downward case. The total change in energy per oscillation is therefore

$$\Delta E = \Delta E_{\text{down}} + \Delta E_{\text{up}} = -\frac{4}{3}\sigma\omega^2 A^3. \tag{45}$$

Remark: The fact that the loss in heat equals the gain in kinetic energy, when an atom joins the moving part of the rope, is a very general result and can easily been understood by looking at things in the frame of the moving rope. In this frame, an atom of mass dm in the heap moves with speed v, and then suddenly comes to rest when it joins the straight part of the rope. The heat loss in the rope's frame (which is the same as the heat loss in the lab frame) is therefore $(dm/2)v^2$, which equals the gain in kinetic energy in the lab frame. The validity of this general result can be traced to the self-evident fact that the speed of the heap with respect to the straight part is the same as the speed of the straight part with respect to the heap. ■

Energy loss per oscillation

The energy of the rope when it has amplitude A is $E = M\omega^2 A^2/2$, thus $dE = M\omega^2 A\, dA$. The number of oscillations in a time dt is $\omega\, dt/2\pi$. Therefore, eq. (45) gives (using $M \approx \sigma L$)

$$\begin{aligned} (\sigma L)\omega^2 A\, dA &= -\left(\frac{\omega\, dt}{2\pi}\right)\left(\frac{4}{3}\sigma\omega^2 A^3\right) \\ \Longrightarrow \qquad \frac{dA}{A^2} &= -\left(\frac{2\omega}{3\pi L}\right) dt. \end{aligned} \tag{46}$$

Integrating this from the start to a time t, and using $A(0) \equiv b$, gives

$$A(t) = \frac{1}{\frac{1}{b} + \frac{2\omega t}{3\pi L}}\,. \tag{47}$$

Remark: For large t, this reduces to

$$A(t) \approx \frac{3\pi L}{2\omega t}\,, \tag{48}$$

which is independent of the initial amplitude b. The $1/t$ behavior implies that the total distance the rope travels barely diverges to infinity, as $t \to \infty$. In terms of $n = \omega t/2\pi$, the number of oscillations undergone, eq. (48) may be written as

$$A(n) \approx \frac{3L}{4n}\,. \qquad \blacksquare \tag{49}$$

The Boston Area Undergraduate Physics Competition

April 21, 2001

Name: ______________________________

School: ______________________________

Year: ______________________________

Address: ______________________________

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e-mail: ______________________________

Phone: ______________________________

Do not turn this page until you are told to do so.

Each of the six problems is worth 10 points.

You have four (4) hours to complete this exam.

Please provide the information requested on this cover sheet. At the end of the exam, hand in this cover sheet with your solutions. You may keep the exam questions.

Show all relevant work in your exam books. Please write neatly. Partial credit will be given for significant progress made toward a correct solution.

You must be enrolled in a full-time undergraduate program to be eligible for prizes.

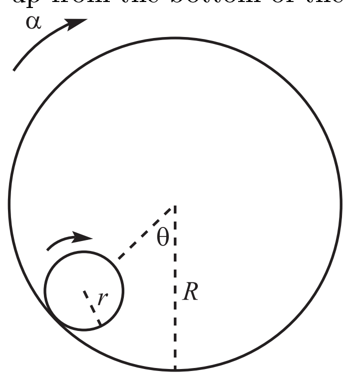
2001

Boston Area Undergraduate Physics Competition

April 21, 2001

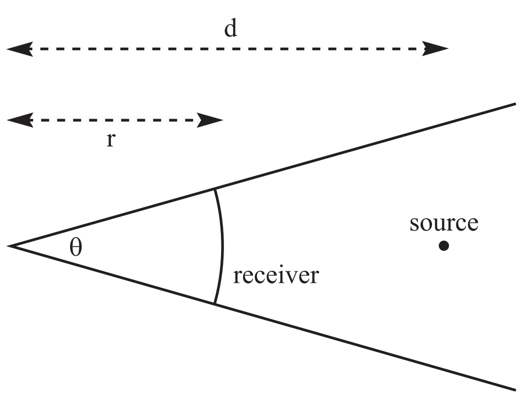
Time: 4 hours

1. A ball (with moment of inertia $I = (2/5)mr^2$) rolls without slipping on the inside of a cylinder of radius R. The cylinder spins around its axis (which points horizontally) with angular acceleration α. What should α be if you wish for the center of the ball to remain motionless at an angle θ up from the bottom of the cylinder (see figure)?

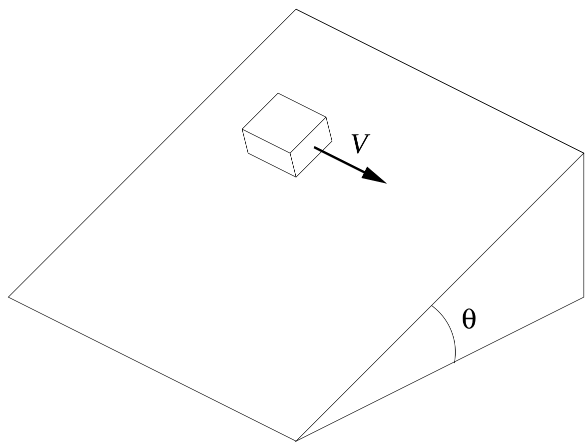


side view

2. A point source emits light (spherically symmetrically) and is located along the axis of a cone (with vertex angle θ), at a distance d from the tip. The inside surface of the cone is reflective. A receiver is located inside the cone. The receiver consists of all points inside the cone that are a distance r from the tip. (See the figure below for a two-dimensional slice of the setup.) What fraction of the light emitted by the source hits the receiver?

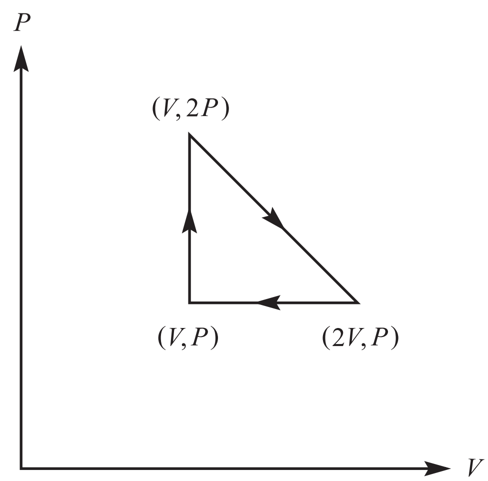


3. A block is placed on a plane inclined at angle θ. The coefficient of friction between the block and plane is $\mu = \tan\theta$. The block is given a kick so that it initially moves with speed V horizontally (i.e., in the direction perpendicular to the direction pointing straight down the plane; see the figure below). What is the speed of the block after a very long time?



4. Find the efficiency of the thermodynamic process shown below. The corners of the triangle are located at the points $(V, 2P)$, (V, P), $(2V, P)$.

   Notes: (1) The "efficiency" of a thermodynamic process is defined to be $\epsilon \equiv (Q_{\text{in}} - Q_{\text{out}})/Q_{\text{in}}$, where Q_{in} and Q_{out} are the quantities of heat added and removed from the system, respectively, (2) the gas is assumed to be an ideal gas, the internal energy of which is $\frac{3}{2}nkT$.



5. A rubber band with initial length L has one end tied to a wall. At $t = 0$, the other end is pulled away from the wall at speed V. (Assume the rubber band stretches uniformly.) At the same time, an ant located at the end not attached to the wall begins to crawl toward the wall, with a speed of u relative to the band. Will the ant reach the wall? If so, how much time will it take?

6. N points in space are connected by a collection of $1\,\Omega$ resistors. The network of resistors is arbitrary, except for the fact that it is "connected" (that is, it is possible to travel between any two points via an unbroken chain of resistors). The number of resistors emanating from any point can be any number from 1 to $N-1$.

   Consider two points that are connected by a $1\,\Omega$ resistor. The network produces an effective resistance between these two points. What is the sum of the effective resistances across all the $1\,\Omega$ resistors in the network?

   Note: You will receive 2 points for stating the correct answer (which must be supported by a few simple examples), and then 8 points for proving the general result.

Boston Area Undergraduate
Physics Competition

Solutions

1. Let the friction force on the ball be F. Then F must cancel the component of gravity in the tangential direction; thus $F = mg\sin\theta$.

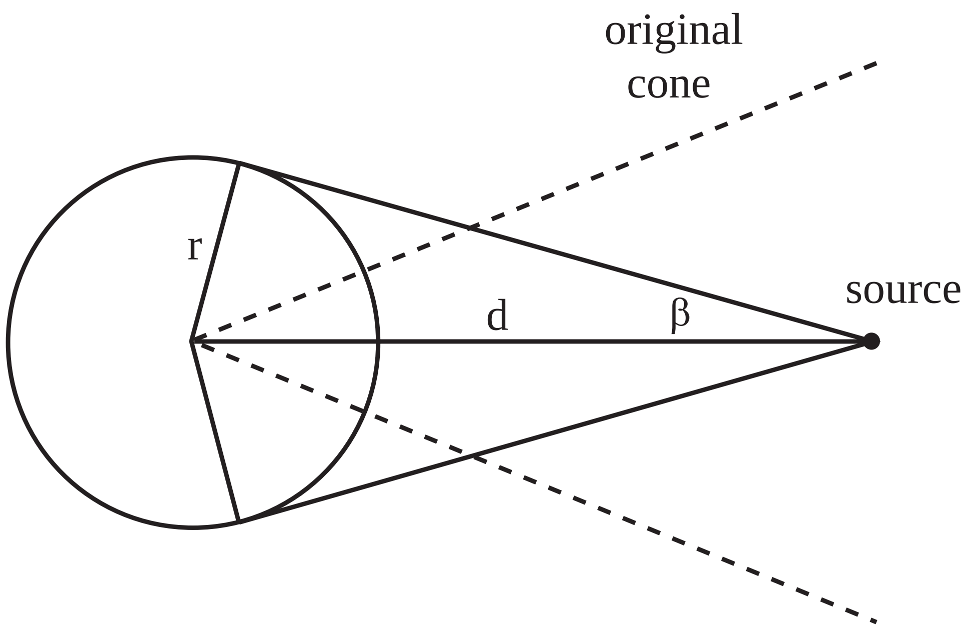
   The torque on the ball is $\tau = Fr$. Using $F = mg\sin\theta$, we get $\tau = mgr\sin\theta$. This torque must equal $I\alpha_b$, where α_b is the angular acceleration of the ball, which is related to the α of the cylinder by $\alpha_b = (R/r)\alpha$. Thus, $\tau = I\alpha_b$ gives

$$mgr\sin\theta = \left(\frac{2}{5}mr^2\right)\left(\frac{R}{r}\alpha\right) \qquad \Longrightarrow \qquad \alpha = \frac{5g\sin\theta}{2R}. \tag{1}$$

2. When the light bounces off the surface, its reflected angle is equal to its incident angle. Therefore, we may use the method of images to determine if a particular ray of light hits the receiver.

   If we look at a two-dimensional cross-section of the cone, we see that when the picture is repeatedly reflected across the edge of the cone, the receiver turns into a full circle of radius r. The method of images therefore tells us that in the original three-dimensional case, the given receiver turns into a full sphere of radius r, centered at the vertex of the cone.

   The given problem therefore reduces to the problem: What fraction of light emitted from a source falls on a sphere of radius r centered at a point a distance d from the source? (Note that the given vertex angle θ is irrelevant.)



From the figure, we see that we need to find the fractional solid angle subtended by a cone with half-angle β, where $\sin\beta = r/d$. Looking at a sphere of radius R, the area of the spherical "cap" subtended by this cone can be found by slicing the cap into circular bands. If α describes the angle away from the top of the cap, then the corresponding circle has radius $2\pi(R\sin\alpha)$, so the resulting integral for the area of the cap is

$$A = \int_0^\beta (2\pi R\sin\alpha)(R\,d\alpha) = -2\pi R^2\cos\alpha\Big|_0^\beta = 2\pi R^2(1-\cos\beta). \quad (2)$$

The fraction of the total area is therefore

$$\text{Fraction} = \frac{A}{4\pi R^2} = \frac{1}{2}(1-\cos\beta) = \frac{1}{2}\left(1-\frac{\sqrt{d^2-r^2}}{d}\right). \quad (3)$$

(If $r > d$, then the fraction equals 1, of course.)

3. The normal force from the plane is $N = mg\cos\theta$, so the friction force is $\mu N = mg\sin\theta$. This force acts in the direction opposite to the motion. There is also the gravitational force of $mg\sin\theta$ pointing down the plane.

The magnitudes of these two forces are equal, so the acceleration along the direction of motion equals the negative of the acceleration in the direction down the plane. Therefore, in a small increment of time, the speed that the block loses along its direction of motion exactly equals the speed that it gains in the direction down the plane. Letting v be the speed of the block, and letting v_y be the component of the speed in the direction down the plane, we therefore have

$$v + v_y = C, \tag{4}$$

where C is a constant. C is given by its initial value, which is $V+0 = V$ (where V is the initial speed of the block). The final value of C is $V_f + V_f = 2V_f$ (where V_f is the final speed of the block), since the block is essentially moving straight down the plane after a very long time. Therefore,

$$2V_f = V \qquad \Longrightarrow \qquad V_f = V/2. \tag{5}$$

4. Start with the first law of thermodynamics (energy conservation),

$$dQ = dU - dW \tag{6}$$

where dQ is an infinitesimal amount of heat added to the system, dU is the change in internal energy, and dW is an infinitesimal amount of mechanical work done on the system.

Consider going around any closed loop in the state of the system. By 'state' we mean pressure p, volume V and temperature T. In our system, knowing any 2 of these determines the third; for instance T is a function of (p, V) which is given by the ideal gas law. U is a function of the state alone, so adding up the dU changes around the closed loop must give zero. Therefore for a single traversal of the loop, $\Delta Q = -\Delta W$. Mechanical work done is caused by changes in volume,

$$dW = -p\,dV, \tag{7}$$

so the integral of dW around a closed loop is just the negative of the area enclosed on the (p, V) plane (for a clockwise loop, as we have). For each traversal of the loop, therefore

$$Q_{\text{in}} - Q_{\text{out}} = \Delta Q = -\Delta W = \text{area enclosed in } (p, V) = \tfrac{1}{2} p_0 V_0. \tag{8}$$

We have given the constants P and V given in the problem the more convenient symbols p_0 and V_0.

We have found the numerator of the fraction giving the efficiency (if you used the methods below to find this numerator, that was fine too). Now all that remains is the denominator, Q_{in}.

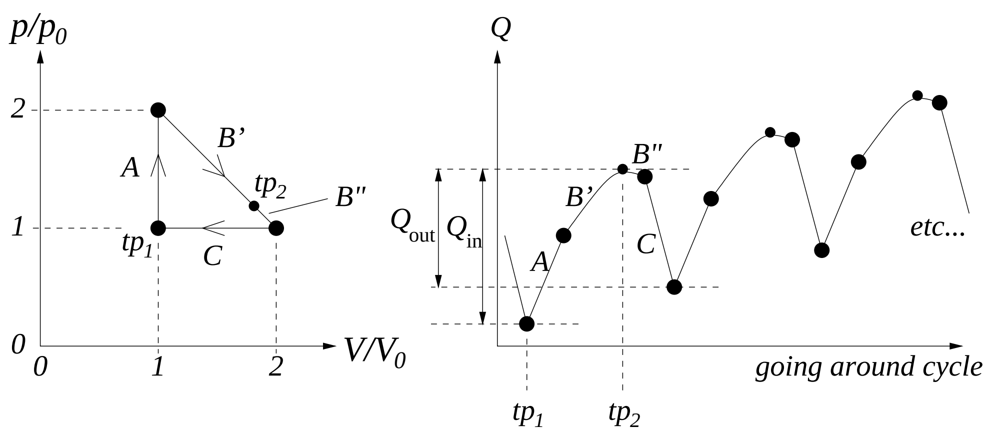
We're given the internal energy $U = \frac{3}{2}nkT$, from which you have to realise that n is the *number of particles* not the number of moles of particles. This is obvious since n is multiplied by k, Boltzmann's constant, rather than R, the gas constant. We have an ideal gas,

$$pV = NRT \quad \text{or equivalently,} \quad pV = nkT, \tag{9}$$

where N is the number of moles, and n the number of particles (note this is swapped from the usual notation for n and N). The second form is the useful one for us, since from it follows that internal energy can be written

$$U = \tfrac{3}{2}pV. \tag{10}$$

Now for any segment of the path we can find ΔU, without explicit reference to T.



We need to consider the complete path, and find the 'turning points' where the flow of heat (sign of dQ) changes direction (tp_1 and tp_2 in the right-hand figure). Q_{in} is then the integral between these turning points along one route of the path (and Q_{out} is the same integral back along the other route). We need to find how Q changes in each side of the triangle, labeled A,B and C (see left-hand figure):

Leg A: no volume change, so $\Delta W = 0$. Due to a doubling of pressure, U has changed from $\frac{3}{2}p_0V_0$ to twice that, so $\Delta Q = \Delta U - \Delta W = +\frac{3}{2}p_0V_0$.

Leg C: no pressure change, so the work integral $\Delta W = -\int p\,dV$ is simply $-p\,\Delta V = p_0V_0$. U goes from its value at the end of leg A back to its original value, so $\Delta U = -\frac{3}{2}p_0V_0$. Together these give $\Delta Q = -\frac{5}{2}p_0V_0$. Checking the signs here was important: the effects combine to give increased heat output.

From the above consideration of dQ in legs A and C, it is clear that one turning point is tp_1 as shown.

Leg B: This is *not* an isothermal change (which would correspond to a hyperbola defined by pV = const, and would imply that $dQ > 0$ everywhere along the leg). It is also tempting to assume that since $\Delta U = 0$ along this leg (same start and end T), one can find ΔW and be done with the problem. Not so! Because T (hence U) is dropping at the end of the leg while W is also dropping (work is being done *by* the system), there is the possibility that dQ changes sign along the leg, making tp_2 happen some fraction of the way along the leg.

We parametrize the leg by a unitless number $x = [1, 2]$, giving $p(x) = p_0(3-x)$ and $V(x) = V_0x$. Therefore

$$dQ = dU - dW = \tfrac{3}{2}(p\,dV + V\,dP) + p\,dV = \frac{p_0V_0}{2}(15 - 8x)\,dx, \quad (11)$$

where the derivatives $dp = -p_0dx$ and $dV = V_0dx$ were used. Clearly dQ changes sign when $15 - 8x = 0$, that is, at $x = 15/8$. Note that this turning point is not simply at $x = 3/2$ (half-way through the leg), when T reaches its maximum. The location of this turning point can also be found by considering the criterion for *adiabaticity* ($dQ = 0$), namely $dp/dV = -\gamma p/V = -\frac{5}{3}p/V$.

Splitting the leg into parts B′ and B″ at this turning point as shown, we need the heat input in leg B′ only,

$$\Delta Q = \int_{x=1}^{x=15/8} dQ = \frac{p_0V_0}{2}\int_{x=1}^{x=15/8}(15 - 8x)\,dx. \quad (12)$$

The x-integral gives $[15x - 4x^2]_1^{15/8} = \frac{49}{16}$ after a little simplification, so $\Delta Q = +\frac{49}{32}p_0V_0$.

Adding ΔQ from legs A and B gives $Q_{\text{in}} = (\frac{3}{2} + \frac{49}{32})p_0 V_0 = \frac{97}{32} p_0 V_0$, an admittedly slightly messy fraction.

Finally the efficiency is

$$\epsilon = \frac{Q_{\text{in}} - Q_{\text{out}}}{Q_{\text{in}}} = \frac{\frac{1}{2}p_0 V_0}{\frac{97}{32}p_0 V_0} = \frac{16}{97} = 0.1649\cdots \text{ or about } 16.5\% \tag{13}$$

Note that this is very close to the incorrect answer of $1/6$ obtained if tp_2 is assumed to be at the lower right vertex.

5. At time t, the movable end of the band is a distance $\ell(t) = L + Vt$ from the wall. Let the ant's distance from the wall be $r(t)$.

   Consider the fraction of the ant's position along the band, $F(t) = r(t)/\ell(t)$. The given question is equivalent to: For what value of t does the fraction, $F(t)$, become zero (if at all)? Let us see how $F(t)$ changes with time.

   After an infinitesimal time, dt, the ant's position, r, increases by $(r/\ell)V\,dt$ due to the stretching, and decreases by $u\,dt$ due to the crawling. Therefore,

$$\begin{aligned} F(t+dt) &= \frac{r + (r/\ell)V\,dt - u\,dt}{\ell + V\,dt} \\ &= \frac{r}{\ell} - \frac{u\,dt}{\ell + V\,dt}. \end{aligned} \tag{14}$$

   To first order in dt, this yields

$$F(t+dt) = F(t) - \frac{u}{\ell}dt. \tag{15}$$

   In other words, $F(t)$ decreases due to the fact that in a time dt the ant crawls a distance $u\,dt$ relative to the band, which has a length approximately $\ell(t)$. Eq. (15) gives

$$\frac{dF(t)}{dt} = -\frac{u}{\ell}. \tag{16}$$

   Using $\ell(t) = L + Vt$ and integrating eq. (16), we obtain

$$F(t) = 1 - \frac{u}{V}\ln\left(1 + \frac{V}{L}t\right), \tag{17}$$

   where the constant of integration has been chosen to satisfy $F(0) = 1$.

We now note that for *any* positive value for u, we can make $F(t) = 0$ by choosing

$$t = \frac{L}{V}\left(e^{V/u} - 1\right). \tag{18}$$

For very large V/u, the time it takes the ant to reach the wall becomes exponentially large, but it does indeed reach it in a finite time.

For very small V/u, (18) reduces to $t \approx L/u$ (using $e^x \approx 1 + x$), as it should.

6. A few simple examples suggest that the answer to the problem is $(N-1)\,\Omega$. Let's prove this in general. (We will use a superposition argument.)

   Consider two points, A and B, that are connected by one of the $1\,\Omega$ resistors. If we put a current I in at A, and take a current I out at B, then the effective resistance between A and B is V/I, where V is the potential difference between the two points.

   The situation where a current I goes in at A and out at B can be considered as the superposition of two setups: (1) Put a current $\frac{N-1}{N}I$ in at A and take a current $\frac{1}{N}I$ out at each of the other $N-1$ points, and (2) Take a current $\frac{N-1}{N}I$ out at B and put a current $\frac{1}{N}I$ in at each of the other $N-1$ points.

   In the first setup, let the current going from A to B be $I^A_{A\to B}$. In the second setup, let the current going from A to B be $I^B_{A\to B}$. (The superscript here denotes the point at which the current of $\frac{N-1}{N}I$ enters or leaves.) Then in the combined setup, the current going from A to B is $I^A_{A\to B} + I^B_{A\to B}$. Since this current passes along a $1\,\Omega$ resistor, the voltage difference between A and B is $V = (I^A_{A\to B} + I^B_{A\to B})(1\,\Omega)$. The effective resistance between A and B is therefore

$$R_{AB} = (I^A_{A\to B} + I^B_{A\to B})(1\,\Omega)/I. \tag{19}$$

   We must now add up these R_{AB} contributions from all of the resistors. Let the desired sum be S. Then

$$S = \sum_{A,B} (I^A_{A\to B} + I^B_{A\to B})(1\,\Omega)/I, \tag{20}$$

   where the sum runs over all pairs of points A, B that are connected by a resistor. By reversing the roles of A and B, we may also write

$$S = \sum_{A,B} (I^B_{B\to A} + I^A_{B\to A})(1\,\Omega)/I. \tag{21}$$

Adding the two previous equations gives

$$2S = \sum_{A,B} (I^A_{A\to B} + I^B_{B\to A})(1\,\Omega)/I + \sum_{A,B} (I^A_{B\to A} + I^B_{A\to B})(1\,\Omega)/I. \quad (22)$$

The first sum is simply the sum of the N currents entering the N points in all of the N setups of type "1" above. Since a current of $\frac{N-1}{N}I$ enters each point (by construction), the first sum equals $(N-1)(1\,\Omega)$. Likewise, the second sum deals with the N currents leaving the N points in all of the N setups of type "2" above, so it also equals $(N-1)(1\,\Omega)$. Eq. (22) therefore gives

$$S = (N-1)\,\Omega. \quad (23)$$

The Boston Area Undergraduate Physics Competition

April 27, 2002

Name: ______________________________

School: ______________________________

Year: ______________________________

Address: ______________________________

____

e-mail: ______________________________

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Do not turn this page until you are told to do so.

You have four (4) hours to complete this exam.

Please provide the information requested on this cover sheet. At the end of the exam, hand in this cover sheet with your solutions. You may keep the exam questions.

Show all relevant work in your exam books. Please write neatly. Partial credit will be given for significant progress made toward a correct solution.

You must be enrolled in a full-time undergraduate program to be eligible for prizes.

2002
Boston Area Undergraduate
Physics Competition

April 27, 2002
Time: 4 hours

Each of the six questions is worth 10 points.

1. Two trapezoidal containers, connected by a tube as shown, hold water.

   (a) If the water in container A is heated (causing it to expand), will water flow through the tube? If so, in which direction?

   (b) If the water in container B is heated (causing it to expand), will water flow through the tube? If so, in which direction?

   (Assume that the containers do not expand when heated.)

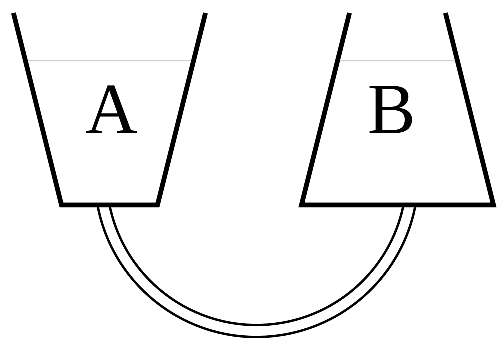


Figure 1: Problem 1, Trapezoidal Containers

2. A mass, which is free to move on a horizontal frictionless plane, is attached to one end of a massless string which wraps partially around a frictionless vertical pole of radius r, as shown on Fig. 2 (top view). You hold on to the other end of the string. At $t = 0$, the mass has speed v_0 in the tangential direction along the dotted circle of radius R shown.

   Your task is to pull on the string so that the mass keeps moving along the dotted circle. You are required to do this in such a way that the string remains in contact with the pole at all times.

   What is the the speed of the mass as a function of time? Explain what happens in this system as time goes by. (Ignore any relativistic effects.)

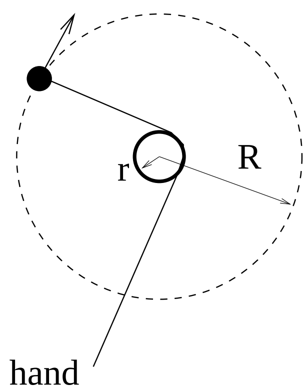


Figure 2: Problem 2, Mass on a String

3. A brick is thrown (from ground level) at an angle θ with respect to the (horizontal) ground. Assume that the long face of the brick remains parallel to the ground at all times, and that there is no deformation in the ground or the brick when the brick hits the ground.

   If the coefficient of friction between the brick and the ground is μ, what should θ be so that the brick travels the maximum total horizontal distance before finally coming to rest?

4. A sheet of metal lies on a roof which is inclined at an angle θ. The coefficient of kinetic friction between the sheet and roof is μ (where $\mu > \tan\theta$).

   During the warmth of daytime, the sheet will expand slightly. And then during the nighttime it will contract. Let the coefficient of thermal expansion of the sheet be α, and let the difference in temperature between day and night be ΔT. Let the length of the sheet (from its upper edge to lower edge) be ℓ.

   How far down the roof will the sheet move in one year if $\theta = 30^\circ$, $\mu = 1$, $\ell = 1\,\text{m}$, $\Delta T = 10^\circ\text{C}$, and $\alpha = 17 \cdot 10^{-6}\,(\text{C}^\circ)^{-1}$ (the α for copper)? Assume uniform contact with the roof.

   (Note: the change in length due to thermal expansion is $\Delta L = \alpha L \Delta T$.)

5. A charged object generally induces an image charge when placed near a metallic plate. If the object moves, currents in the metal will lead to damping of its motion. Consider the following model for the dissipation: The image charge's motion lags by time τ behind the object's motion.

   (a) What applied force is necessary to sustain the motion of an object with charge q moving with constant velocity $\mathbf{v}$ parallel to an infinite metal plate, characterized by the time lag τ? Let the distance from the plate be r.

   (b) Find the leading contribution (for small v and τ) to the force calculated in part (a) in the direction of $\mathbf{v}$. What is the damping coefficient γ (which is defined by $\mathbf{F} = -\gamma\mathbf{v}$)?

   (c) What is the damping coefficient for motion perpendicular to the plate?

6. A small ball is attached to a massless string of length L, the other end of which is attached to a very thin vertical pole. The ball is thrown so that it initially travels in a horizontal circle, with the string making an angle θ_0 with the vertical.

   As time goes on, the string will wrap itself around the pole. Assume that (1) the pole is thin enough so that the length of string in the air decreases very slowly, so that the ball's motion may always be approximated as a horizontal circle, and (2) the pole has enough friction so that the string does not slide on the pole, once it touches it.

   Find the difference in height between the point where the string is attached to the pole and the point where the ball eventually hits the pole.

   Also, find the ratio of the ball's final speed (right before it hits the pole) to its initial speed.

Solutions

8th Annual
Boston Area Undergraduate
Physics Competition

April 27, 2002

1. (a) If A is heated, then water will flow from B to A. The reason can be seen as follows. The pressure at depth h is given by $P = \rho g h$. When the water in A expands, the height h increases, but the density ρ decreases. What happens to the product ρh? The density goes like $1/A$, where A is the area of the trapezoidal cross section. But $A = wh$, where w is the width at half height. Therefore, $P = \rho g h \propto h/A = 1/w$. And since w increases as the water level rises, the pressure in A decreases, and water flows from B to A.

   (b) If B is heated, then water will again flow from B to A. The same reasoning used above works here, except than now the w in container B *decreases*, so that the pressure in B *increases*, so that the water again flows from B to A.

2. Let F be the tension in the string. The angle (at the mass) between the string and the radius of the dotted circle is $\theta = \sin^{-1}(r/R)$. In terms of θ, the radial and tangential $F = ma$ equations are

$$\begin{aligned} F\cos\theta &= mv^2/R, \qquad \text{and} \\ F\sin\theta &= m\dot{v}. \end{aligned} \tag{1}$$

Solving for F in the second equation and substituting into the first gives

$$\frac{m\dot{v}\cos\theta}{\sin\theta} = \frac{mv^2}{R}\,. \tag{2}$$

Separating variables and integrating gives

$$\begin{aligned} \int_{v_0}^{v} \frac{dv}{v^2} &= \frac{\tan\theta}{R}\int_0^t dt \\ \Longrightarrow \quad \frac{1}{v_0} - \frac{1}{v} &= \frac{\tan\theta}{R} t \\ \Longrightarrow \quad v &= \left(\frac{1}{v_0} - \frac{\tan\theta}{R} t\right)^{-1}. \end{aligned} \tag{3}$$

Note that v becomes infinite when

$$t = T \equiv \frac{R}{v_0 \tan\theta}\,. \tag{4}$$

In other words, you can keep the mass moving in the desired circle only up to time T. After that, it is impossible. (Of course, it will become impossible, for all practical purposes, long before v becomes infinite.)

The total distance, $d = \int v\, dt$, is infinite, because this integral (barely) diverges (like a log), as t approaches T.

3. Let V be the initial speed. The horizontal speed and initial vertical speed are then $V\cos\theta$ and $V\sin\theta$, respectively. You can easily show that the distance traveled in the air is the standard

$$d_{\text{air}} = \frac{2V^2\sin\theta\cos\theta}{g}\,. \tag{5}$$

To find the distance traveled along the ground, we must determine the horizontal speed just after the impact has occurred. The normal force, N, from the ground is what reduces the vertical speed from $V\sin\theta$ to zero, during the impact. So we have

$$\int N\,dt = mV\sin\theta, \tag{6}$$

where the integral runs over the time of the impact. But this normal force (when multiplied by μ, to give the horizontal friction force) also produces a sudden decrease in the horizontal speed, during the time of the impact. So we have

$$m\Delta v_x = -\int(\mu N)\,dt = -\mu mV\sin\theta \qquad \Longrightarrow \qquad \Delta v_x = -\mu V\sin\theta. \tag{7}$$

(We have neglected the effect of the mg gravitational force during the short time of the impact, since it is much smaller than the N impulsive force.) Therefore, the brick begins its sliding motion with speed

$$v = V\cos\theta - \mu V\sin\theta. \tag{8}$$

Note that this is true only if $\tan\theta \le 1/\mu$. If θ is larger than this, then the horizontal speed simply becomes zero, and the brick moves no further. (Eq. (8) would give a negative value for v.)

The friction force from this point on is μmg, so the acceleration is $a = -\mu g$. The distance traveled along the ground can easily be shown to be

$$d_{\text{ground}} = \frac{(V\cos\theta - \mu V\sin\theta)^2}{2\mu g}\,. \tag{9}$$

We want to find the angle that maximizes the total distance, $d_{\text{total}} = d_{\text{air}} + d_{\text{ground}}$. From eqs. (5) and (9) we have

$$\begin{aligned} d_{\text{total}} &= \frac{V^2}{2\mu g}\left(4\mu\sin\theta\cos\theta + (\cos\theta - \mu\sin\theta)^2\right) \\ &= \frac{V^2}{2\mu g}(\cos\theta + \mu\sin\theta)^2. \end{aligned} \tag{10}$$

Taking the derivative with respect to θ, we see that the maximum total distance is achieved when

$$\tan\theta = \mu. \tag{11}$$

Note, however, that the above analysis is valid only if $\tan\theta \le 1/\mu$ (from the comment after eq. (8)). We therefore see that if:

- $\mu < 1$, then the optimal angle is given by $\tan\theta = \mu$. (The brick continues to slide after the impact.)

- $\mu \geq 1$, then the optimal angle is $\theta = 45°$. (The brick stops after the impact, and $\theta = 45°$ gives the maximum value for the d_{air} expression in eq. (5).)

4. The key point in this problem is that the sheet expands about a certain stationary point, but contracts around another (so that it ends up moving down the roof like an inchworm). We must find the locations of these two points.

   Let's consider the expansion first. Let the stationary point be a distance a from the top and b from the bottom (so $a+b=\ell$). The lower part of the sheet, of mass $m(b/\ell)$, will be moving downward along the roof. Therefore, it will feel a friction force upward, with magnitude $\mu N = \mu m(b/\ell)g\cos\theta$. Likewise, the upper part, of mass $m(a/\ell)$, will feel a friction force downward, with magnitude $\mu m(a/\ell)g\cos\theta$.

   Because the sheet is not accelerating, the difference in these two friction forces must equal the downward force of gravity along the roof, namely $mg\sin\theta$. Therefore,

$$\mu m\left(\frac{b-a}{\ell}\right)g\cos\theta = mg\sin\theta$$
$$\Longrightarrow \quad b-a = \frac{\ell\tan\theta}{\mu}. \tag{12}$$

   Note that this implies $b > a$. Also note that $b-a$ of course cannot be greater than ℓ; therefore, if $\tan\theta > \mu$, then there are no solutions for a and b, so the forces cannot balance, and so the sheet will accelerate down the roof. (This $\tan\theta > \mu$ result is a general result, of course, for the equilibrium of an object on an inclined plane.)

   When the object contracts, all of the above analysis holds, except that now the roles of a and b are reversed. The stationary point is now closer to the bottom. With a and b defined in the same way as above, we find (as you can verify)

$$a-b = \frac{\ell\tan\theta}{\mu}. \tag{13}$$

   Putting eqs. (12) and (13) together, we see that the stationary points of expansion (P_{e}) and contraction (P_{c}) are separated by a distance

$$d = \frac{\ell\tan\theta}{\mu}. \tag{14}$$

   During the expansion, the point P_{c} moves downward a distance

$$\epsilon = \alpha d\,\Delta T = \frac{\alpha\ell\tan\theta\Delta T}{\mu}. \tag{15}$$

   and then during the contraction it remains fixed. (Equivalently, the center of the sheet moves downward by a distance of half this, for both the expansion and contraction.) Therefore, during one complete cycle (that is, during a span of 24 hours), the sheet moves downward by the distance ϵ given above.

   Plugging in the given numbers, we see that the distance the sheet moves in one year is given by

$$(365)\epsilon = \frac{(365)(17\cdot 10^{-6}(\text{C}°)^{-1})(1\,\text{m})(\tan 30°)(10°\text{C})}{1} \approx 0.036\,\text{m} = 3.6\,\text{cm}. \tag{16}$$

5. (a) The image charge lags behind the given charge by a distance $v\tau$. Therefore, from the pythagorean theorem, the separation between the two charges is $d = \sqrt{(2r)^2 + (v\tau)^2}$. The force necessary to maintain constant motion (parallel to the plate) is the negative of the Coulomb force between the charges. Hence, the desired force is

$$F = \frac{kq^2}{d^2} = \frac{kq^2}{4r^2 + v^2\tau^2}\,. \tag{17}$$

This force points at an angle of θ with respect to the normal to the plate, where θ is given by

$$\tan\theta = \frac{v\tau}{2r}\,. \tag{18}$$

(b) The component of the above force in the direction of $\mathbf{v}$ is

$$F_v \equiv F\sin\theta = \frac{kq^2}{4r^2 + v^2\tau^2}\left(\frac{v\tau}{\sqrt{4r^2 + v^2\tau^2}}\right). \tag{19}$$

To first order in the small quantity $v\tau$, we may neglect the $v\tau$ terms in the denominator. Therefore,

$$F_v \approx \frac{kq^2 v\tau}{8r^3}\,. \tag{20}$$

This is the force necessary to overcome the damping force, $\mathbf{F} = -\gamma\mathbf{v}$. So we see that

$$\gamma = \frac{kq^2\tau}{8r^3}\,. \tag{21}$$

(c) For motion perpendicular to the plate, the lagging motion of the image charge implies that the charges will be a distance $2r + v\tau$ apart. The force between them is therefore

$$F = \frac{kq^2}{(2r + v\tau)^2} \approx \frac{kq^2}{4(r^2 + rv\tau)} \approx \frac{kq^2(r^2 - rv\tau)}{4r^4} = \frac{kq^2}{4r^2} - \frac{kq^2 v\tau}{4r^3}\,. \tag{22}$$

We see that the attractive force is slightly less than it would be if v were zero. This is due to the damping force, $\mathbf{F} = -\gamma\mathbf{v}$, where

$$\gamma = \frac{kq^2\tau}{4r^3}\,. \tag{23}$$

6. Let ℓ and θ be the length of the string and the angle it makes with the pole, respectively, as functions of time.

The two facts we will need to solve this problem are: (1) the radial $F = ma$ equation, and (2) the conservation of energy statement.

Approximating the motion at any time by a horizontal circle (of radius $\ell\sin\theta$), we see that the vertical force applied by the string is mg, and hence the horizontal force is $mg\tan\theta$. Therefore, the radial $F = ma$ equation is

$$\frac{mv^2}{\ell\sin\theta} = mg\tan\theta. \tag{24}$$

Conservation of energy says that the change in KE plus the change in PE is zero. We'll write the change in KE simply as $d(mv^2/2)$ for now. We claim that the change in PE is given by $mg\ell \sin\theta\, d\theta$. This can be seen as follows.

Put a mark on the string a small distance $d\ell$ down from the contact point. After a short time, this mark will become the contact point. The height of this mark will *not* change (to first order, at least) during this process. This is true because initially the mark is a height $\ell\cos\theta$ below the initial contact point. And it is still (to first order) this far below the initial contact point when the mark becomes the contact point, because the angle is still very close to θ, so any errors will be of order $d\ell\, d\theta$.

The change in height of the ball relative to this mark (whose height is essentially constant) is due to the $\ell - d\ell$ length of string in the air "swinging" up through an angle $d\theta$. Multiplying by $\sin\theta$ to obtain the vertical component of this arc, we see that the change in height is $((\ell - d\ell)d\theta)\sin\theta$. This equals $\ell\sin\theta\, d\theta$, to first order, as was to be shown.

Therefore, conservation of energy gives

$$\frac{1}{2}d(mv^2) + mg\ell\sin\theta\, d\theta = 0. \tag{25}$$

We will now use eqs. (24) and (25) to solve for ℓ in terms of θ. Substituting the v^2 from eq. (24) into eq. (25) gives

$$\begin{aligned}
& d(\ell\sin\theta\tan\theta) + 2\ell\sin\theta\, d\theta = 0 \\
\Longrightarrow\quad & (d\ell\sin\theta\tan\theta + \ell\cos\theta\tan\theta\, d\theta + \ell\sin\theta\sec^2\theta) + 2\ell\sin\theta\, d\theta = 0 \\
\Longrightarrow\quad & d\ell\frac{\sin^2\theta}{\cos\theta} + 3\ell\sin\theta\, d\theta + \ell\frac{\sin\theta}{\cos^2\theta} = 0 \\
\Longrightarrow\quad & \int\frac{d\ell}{\ell} = -\int\frac{3\cos\theta\, d\theta}{\sin\theta} - \int\frac{d\theta}{\sin\theta\cos\theta} \\
\Longrightarrow\quad & \ln\ell = -3\ln(\sin\theta) + \ln\left(\frac{\cos\theta}{\sin\theta}\right) + C \\
\Longrightarrow\quad & \ell = A\frac{\cos\theta}{\sin^4\theta}, \qquad \text{where} \quad A = L\left(\frac{\sin^4\theta_0}{\cos\theta_0}\right)
\end{aligned} \tag{26}$$

is determined from the initial condition, $\ell = L$ when $\theta = \theta_0$. Note that this result implies that $\theta = \pi/2$ when the ball hits the pole (that is, when $\ell = 0$). The last integral in the fourth line above can be found in various ways. One is to multiply by $\cos\theta/\cos\theta$, and then note that $d\theta/\cos^2\theta = d(\tan\theta)$.

Now let's find the position where the ball hits the pole. The vertical distance a small piece of the string covers is $dy = d\ell\cos\theta$. So the ball hits the pole at a y value (relative to the top) given by

$$y = \int d\ell\,\cos\theta = A\int d\left(\frac{\cos\theta}{\sin^4\theta}\right)\cos\theta, \tag{27}$$

where the integral runs from θ_0 to $\pi/2$, and A is given in eq. (26). We may now integrate by parts to obtain

$$\frac{y}{A} \;=\; \left(\frac{\cos\theta}{\sin^4\theta}\right)\cos\theta - \int\left(\frac{\cos\theta}{\sin^4\theta}\right)(-\sin\theta)\, d\theta$$

$$
\begin{aligned}
&= \frac{\cos^2\theta}{\sin^4\theta} + \int \frac{\cos\theta}{\sin^3\theta}\,d\theta \\
&= \left(\frac{\cos^2\theta}{\sin^4\theta} - \frac{1}{2\sin^2\theta}\right)\Bigg|_{\theta_0}^{\pi/2}. \qquad (28)
\end{aligned}
$$

Using the value of A given in eq. (26), we obtain

$$
\begin{aligned}
y &= L\left(\frac{\sin^4\theta_0}{\cos\theta_0}\right)\left(-\frac{1}{2} - \left(\frac{\cos^2\theta_0}{\sin^4\theta_0} - \frac{1}{2\sin^2\theta_0}\right)\right) \\
&= L\left(\frac{\sin^4\theta_0}{\cos\theta_0}\right)\left(-\frac{\cos^2\theta_0}{\sin^4\theta_0} + \frac{\cos^2\theta_0}{2\sin^2\theta_0}\right) \\
&= -L\cos\theta_0\left(1 - \frac{\sin^2\theta_0}{2}\right). \qquad (29)
\end{aligned}
$$

Since the ball starts at a position $y = -L\cos\theta_0$, we see that it rises up a distance $\Delta y = (1/2)L\cos\theta_0\sin^2\theta_0$ during the course of its motion. (This change in height happens to be maximum when $\tan\theta_0 = \sqrt{2}$, in which case $\Delta y = L/3\sqrt{3}$.)

By conservation of energy, we can find the final speed from

$$
\frac{1}{2}mv_f^2 = \frac{1}{2}mv_i^2 - mg\left(\frac{1}{2}L\cos\theta_0\sin^2\theta_0\right) \qquad (30)
$$

From eq. (24), we have

$$
v_i^2 = gL\frac{\sin^2\theta_0}{\cos\theta_0}\,. \qquad (31)
$$

Therefore,

$$
\begin{aligned}
\frac{1}{2}mv_f^2 &= \frac{1}{2}mgL\frac{\sin^2\theta_0}{\cos\theta_0} - \frac{1}{2}mgL\cos\theta_0\sin^2\theta_0 \\
&= \frac{1}{2}mgL\sin^2\theta_0\left(\frac{1}{\cos\theta_0} - \cos\theta_0\right) \\
&= \frac{1}{2}mgL\frac{\sin^4\theta_0}{\cos\theta_0}\,. \qquad (32)
\end{aligned}
$$

Hence,

$$
v_f^2 = gL\frac{\sin^4\theta_0}{\cos\theta_0}\,. \qquad (33)
$$

Combining eqs. (31) and (33), we finally have

$$
\frac{v_f}{v_i} = \sin\theta_0. \qquad (34)
$$

2003

Boston Area Undergraduate

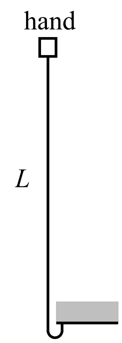
Physics Competition

April 26, 2003

Time: 4 hours

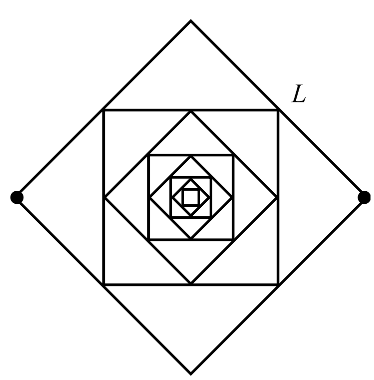
Each of the six questions is worth 10 points.

1. A rope with mass M and length L is held in the position shown below, with one end attached to a support. (Assume that only a negligible length of the rope starts out below the support.) The rope is released. Find the force that the support applies to the rope, as a function of time.



2. A mass is connected to one end of a massless string, the other end of which is connected to a very thin frictionless vertical pole. The string is initially wound completely around the pole (in a very large number of little horizontal circles), with the mass touching the pole. The mass is released, and the string gradually unwinds. What angle does the string make with the pole at the moment it becomes completely unwound?

3. Consider building the following resistor circuit. Start with a square of side length L. Connect the centers of each side to form another square. Connect the centers of each side of this square to form yet another square, and so on, to infinity. What is the resistance between two opposite corners of the original square? (Assume that all the wires in the circuit have the same cross-section and resistivity.) Give your answer in terms of the resistance, R, of a length L of the wire.



4. A charged particle travels into a region in which there is a friction force proportional to the particle's velocity. It travels 10 cm from the point of entry into the region before coming to rest. If a magnetic field of unknown strength is turned on in the region, the particle instead moves along a spiral and comes to rest 6 cm from the point of entry. (This 6 cm is the magnitude of the net displacement, that is, the straight-line distance.) How far from the point of entry will the particle come to rest if the magnetic field is doubled?

5. Consider a ball (with moment of inertia $I = (2/5)MR^2$) which bounces elastically off a surface. Assume that the ball's speed in the direction perpendicular to the surface is the same before and after a bounce. Also, assume that the ball is made of a type of rubber which allows it to not slip on the surface (which has friction) during the bounce. (This implies that the angular and linear motions may affect each other.)

   The ball is projected from the surface of a plane which is inclined at angle θ. The initial velocity of the ball is perpendicular to the plane and has magnitude V. The initial angular velocity is zero. Find the component of the ball's velocity along the plane, immediately after the nth bounce.

6. A ball (with moment of inertia $I = (2/5)MR^2$) rolls without slipping on the inside surface of a fixed cone, whose tip points downward. The half-angle at the vertex of the cone is θ. Initial conditions have been set up so that the ball travels around the cone in a horizontal circle of radius ℓ, with the contact points (the points on the ball that touch the cone) tracing out a given circle (not necessarily a great circle) on the ball.

   What should the radius of the circle of these contact points be, if you want the sphere to travel around the cone as fast as possible? (You may work in the approximation where R is much less than ℓ. Also, assume that the coefficient of friction between the ball and the cone is arbitrarily large.)

Solutions

9th Annual
Boston Area Undergraduate
Physics Competition

April 26, 2003

1. At time t, the free end of the rope is moving at speed gt, and it has fallen a distance $gt^2/2$. This distance gets "doubled up" below the support. So at time t, a length $gt^2/4$ is hanging at rest, and a length $L - gt^2/4$ is moving at speed gt. The momentum of the entire rope is therefore $p = -\rho(L - gt^2/4)(gt)$, where $\rho \equiv M/L$ is the mass density, and the minus sign signifies downward motion.

   The forces on the entire rope are $Mg = \rho Lg$ downward, and N (the force from the support) upward. $F = dp/dt$ applied to the entire rope therefore gives

$$\begin{aligned} N - \rho Lg &= \frac{d}{dt}\left(-\rho Lgt + \frac{\rho g^2 t^3}{4}\right) \\ \Longrightarrow \qquad N &= \frac{3\rho g^2 t^2}{4}\,. \end{aligned} \tag{1}$$

   (Note that this equals $3\rho v^2/4$.) This result holds until $t = \sqrt{4L/g}$, which is the time it takes the free end to fall a distance $2L$. After this time, the force from the support is simply the weight of the entire rope, Mg.

   Remark: At time t, the part of the rope that is hanging at rest has weight $\rho(gt^2/4)g = \rho g^2 t^2/4$. From eq. (1), we see that the support must apply a force that is three times the weight of this motionless part of the rope. The extra force is necessary because the support must do more than hold up the motionless part. It must cause the change in momentum of the atoms in the rope that are abruptly brought to rest from their freefall motion.

2. Because the pole is very thin, we can approximate the motion of the mass at all times by a circle. Let θ be the desired angle of the string when it becomes completely unwound. Then the total change in height of the mass is $L\cos\theta$. So conservation of energy gives

$$mgL\cos\theta = \frac{mv^2}{2}\,. \tag{2}$$

   The vertical component of the tension in the string is essentially mg, because the height of the mass changes so slowly (because the pole is so thin). Therefore, the horizontal component of the tension is $mg\tan\theta$. The mass travels in a horizontal circle of radius $L\sin\theta$, so the horizontal $F = ma$ equation gives

$$mg\tan\theta = \frac{mv^2}{L\sin\theta}\,. \tag{3}$$

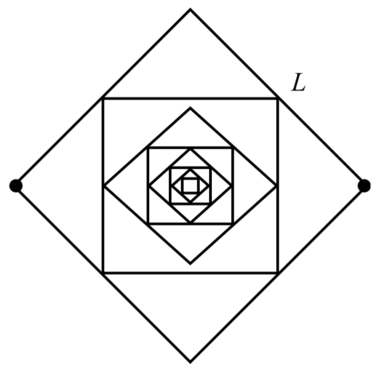
   Dividing eq. (3) by eq. (2) gives

$$\tan\theta = \sqrt{2}. \tag{4}$$

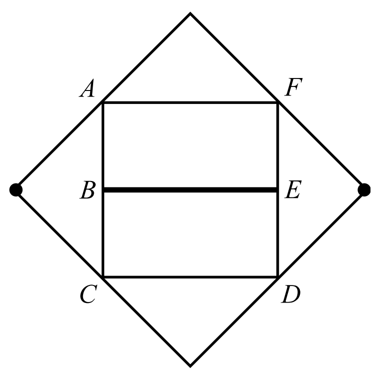
   The numerical value turns out to be $\theta \approx 54.7°$.

3. Let the desired resistance between opposite corners be xR, where x is a numerical factor to be determined. Consider all the squares except the largest two. Label this subset of resistors as S_3. S_3 is identical to the original circuit, except that it is shrunk by a factor of 2. Therefore, the resistance between the right and left corners of S_3 is $xR/2$ (because all the cross sections are the same, and resistance is proportional to length).

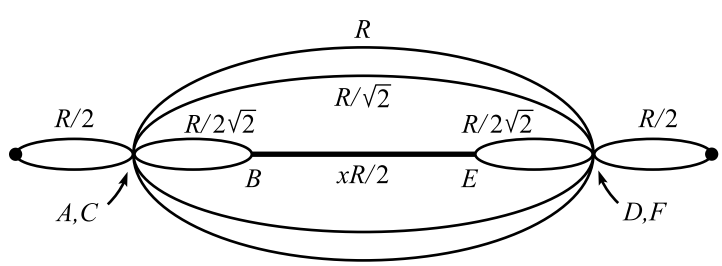
   From left-right symmetry, all the points on the vertical bisector of the circuit below are at the same potential. Therefore, we can separate the top and bottom corners of S_3 from the horizontal lines they touch, as shown.



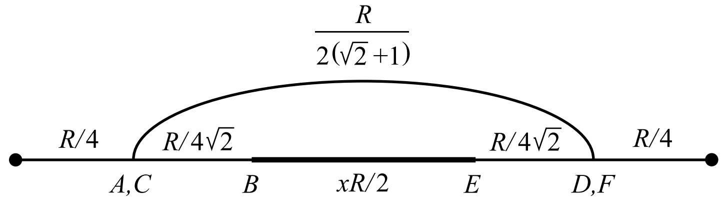
   We can then think of S_3 as an effective resistor of resistance $xR/2$, as shown.



   We can now simplify this circuit by noting the top-bottom symmetry, which tells us that we can identify A with C, and also D with F. (Note that we *cannot* identify B with A, C; or E with D, F.) We arrive at:



Reducing things a bit more gives:



Reducing one last time, and setting the result equal to xR, gives:

$$xR = \frac{R}{2} + \left(\frac{1}{\frac{R}{2(\sqrt{2}+1)}} + \frac{1}{\frac{R}{2\sqrt{2}} + \frac{xR}{2}} \right)^{-1} \tag{5}$$

We must now solve for x. Gradual simplification gives:

$$2x - 1 = \frac{1}{\sqrt{2} + 1 + \frac{\sqrt{2}}{\sqrt{2}x+1}}$$

$$\Longrightarrow \qquad (2x-1)\left((2+\sqrt{2})x + 2\sqrt{2} + 1\right) = \sqrt{2}x + 1$$

$$\Longrightarrow \qquad (2+\sqrt{2})x^2 + \sqrt{2}x - (\sqrt{2}+1) = 0$$

$$\Longrightarrow \qquad x^2 + (\sqrt{2}-1)x - \frac{1}{\sqrt{2}} = 0$$

$$\Longrightarrow \qquad x = \frac{1}{2}(\sqrt{3} - \sqrt{2} + 1) \approx 0.659. \tag{6}$$

4. Integrating $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\Delta\mathbf{p} = \int \mathbf{F}\, dt. \tag{7}$$

Before the magnetic field is turned on, the force on the particle takes the form, $\mathbf{F} = -b\mathbf{v}$. Therefore,

$$\Delta\mathbf{p} = \int (-b\mathbf{v})\, dt. \tag{8}$$

But $\int \mathbf{v}\, dt = \Delta\mathbf{x}$, where $\Delta\mathbf{x}$ is the total displacement (which equals 10 cm straight into the region here). So we have

$$\Delta\mathbf{p}_0 = -b\Delta\mathbf{x}_0, \tag{9}$$

where the subscript denotes the zero-$\mathbf{B}$ case.

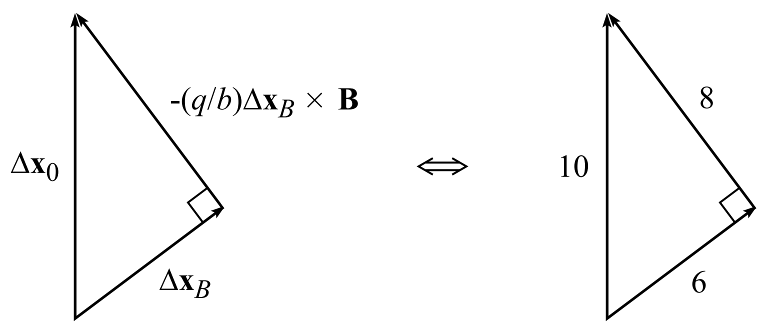
Let us now turn on the magnetic field. We have $\mathbf{F} = -b\mathbf{v} + q\mathbf{v} \times \mathbf{B}$, so

$$\begin{aligned} \Delta\mathbf{p} &= \int (-b\mathbf{v} + q\mathbf{v} \times \mathbf{B})\, dt \\ &= -b\Delta\mathbf{x} + q\Delta\mathbf{x} \times \mathbf{B}. \end{aligned} \tag{10}$$

But the $\Delta \mathbf{p}$ here is the same as the $\Delta \mathbf{p}_0$ in eq. (9), because the the particle has the same initial velocity and the same final velocity (namely zero). Therefore, we have (after dividing through by $-b$)

$$\Delta \mathbf{x}_0 = \Delta \mathbf{x}_B - (q/b)\Delta \mathbf{x}_B \times \mathbf{B}. \tag{11}$$

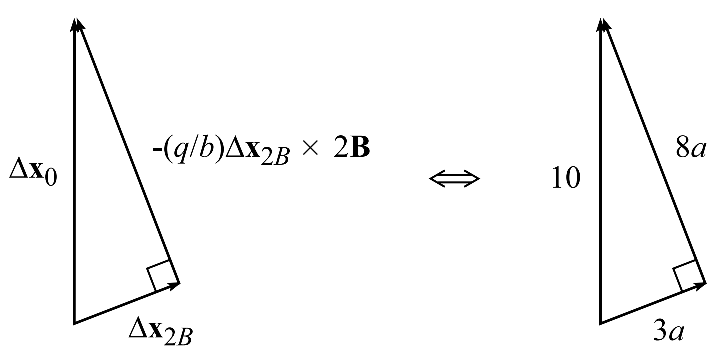
The two terms on the righthand side represent orthogonal vectors. Since the sum of these two orthogonal vectors equals the vector on the lefthand side, and since $|\Delta \mathbf{x}_B|/|\Delta \mathbf{x}_0| = 6/10$, we see that we have the following 6-8-10 right triangle.



If we now double the magnetic field, we have

$$\Delta \mathbf{x}_0 = \Delta \mathbf{x}_{2B} - (q/b)\Delta \mathbf{x}_{2B} \times 2\mathbf{B}. \tag{12}$$

The ratio of the magnitudes of the two vectors on the righthand side is twice the ratio of the two vectors in eq. (11). That is, it is 8/3 instead of 8/6. So we now have the following right triangle.



The Pythagorean theorem gives the value of a as $10/\sqrt{73}$. Therefore, the net displacement of the particle is $|\Delta \mathbf{x}_{2B}| = 3a = 30/\sqrt{73} \approx 3.51\,\text{cm}$.

5. Let us ignore the tilt of the plane for a moment and determine how the ω_f and v_f after a bounce are related to the ω_i and v_i before the bounce (where v denotes the velocity component parallel to the plane). Let the positive directions of velocity and force be to the right along the plane, and let the positive direction of angular velocity

be counterclockwise. If we integrate (over the small time of a bounce) the friction force and the resulting torque, we obtain

$$\begin{aligned} F &= \frac{dp}{dt} \quad \Longrightarrow \quad \int F\,dt = \Delta p, \\ \tau &= \frac{dL}{dt} \quad \Longrightarrow \quad \int \tau\,dt = \Delta L. \end{aligned} \tag{13}$$

But $\tau = RF$. And since R is constant, we have

$$\Delta L = \int RF\,dt = R\int F\,dt = R\Delta p. \tag{14}$$

Therefore,

$$I(\omega_f - \omega_i) = Rm(v_f - v_i). \tag{15}$$

But conservation of energy gives

$$\begin{aligned} \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 &= \frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 \\ \Longrightarrow \qquad I(\omega_f^2 - \omega_i^2) &= m(v_i^2 - v_f^2). \end{aligned} \tag{16}$$

Dividing this equation by eq. (15) gives[1]

$$R(\omega_f + \omega_i) = -(v_f + v_i). \tag{17}$$

We can now combine this equation with eq. (15), which can be rewritten (using $I = (2/5)mR^2$) as

$$\frac{2}{5}R(\omega_f - \omega_i) = v_f - v_i. \tag{18}$$

Given v_i and ω_i, the previous two equations are two linear equations in the two unknowns, v_f and ω_f. Solving for v_f and ω_f, and then writing the result in matrix notation, gives

$$\begin{pmatrix} v_f \\ R\omega_f \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & -4 \\ -10 & -3 \end{pmatrix}\begin{pmatrix} v_i \\ R\omega_i \end{pmatrix} \equiv \mathcal{A}\begin{pmatrix} v_i \\ R\omega_i \end{pmatrix}. \tag{19}$$

Note that

$$\mathcal{A}^2 = \frac{1}{49}\begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix} = \mathcal{I}. \tag{20}$$

Let us now consider the effects of the tilted plane. Since the ball's speed perpendicular to the plane is unchanged by each bounce, the ball spends the same amount of time in the air between any two successive bounces. This time equals $T = 2V/g\cos\theta$, because the component of gravity perpendicular to the plane is $g\cos\theta$. During this time, the speed along the plane increases by $(g\sin\theta)T = 2V\tan\theta \equiv V_0$.

[1] We have divided out the trivial $\omega_f = \omega_i$ and $v_f = v_i$ solution, which corresponds to slipping motion on a frictionless plane. The nontrivial solution we will find shortly is the non-slipping one. Basically, to conserve energy, there must be no work done by friction. But since work is force times distance, this means that either the plane is frictionless, or that there is no relative motion between ball's contact point and the plane. Since we are given that the plane has friction, the latter (non-slipping) case must be the one we are concerned with.

Let $\mathbf{Q}$ denote the $(v, R\omega)$ vector at a given time (where v denotes the velocity component parallel to the plane). The ball is initially projected with $\mathbf{Q} = \mathbf{0}$. Therefore, right before the first bounce, we have $\mathbf{Q}_1^{\textbf{before}} = (V_0, 0) \equiv \mathbf{V}_0$. (We have used the fact that ω doesn't change while the ball is in the air.) Right after the first bounce, we have $\mathbf{Q}_1^{\textbf{after}} = \mathcal{A}\mathbf{V}_0$. We then have $\mathbf{Q}_2^{\textbf{before}} = \mathcal{A}\mathbf{V}_0 + \mathbf{V}_0$, and so $\mathbf{Q}_2^{\textbf{after}} = \mathcal{A}(\mathcal{A}\mathbf{V}_0 + \mathbf{V}_0)$. Continuing in this manner, we see that

$$\begin{aligned} \mathbf{Q}_n^{\textbf{before}} &= (\mathcal{A}^{n-1} + \cdots + \mathcal{A} + \mathcal{I})\mathbf{V}_0, \quad \text{and} \\ \mathbf{Q}_n^{\textbf{after}} &= (\mathcal{A}^{n} + \cdots + \mathcal{A}^2 + \mathcal{A})\mathbf{V}_0. \end{aligned} \tag{21}$$

However, $\mathcal{A}^2 = \mathcal{I}$, so all the even powers of $\mathcal{A}$ equal $\mathcal{I}$. The value of $\mathbf{Q}$ after the nth bounce is therefore given by

$$\begin{aligned} n \text{ even} \quad \Longrightarrow \quad \mathbf{Q}_n^{\textbf{after}} &= \frac{n}{2}(\mathcal{A} + \mathcal{I})\mathbf{V}_0. \\ n \text{ odd} \quad \Longrightarrow \quad \mathbf{Q}_n^{\textbf{after}} &= \frac{1}{2}\Big((n+1)\mathcal{A} + (n-1)\mathcal{I}\Big)\mathbf{V}_0. \end{aligned} \tag{22}$$

Using the value of $\mathcal{A}$ defined in eq. (19), we find

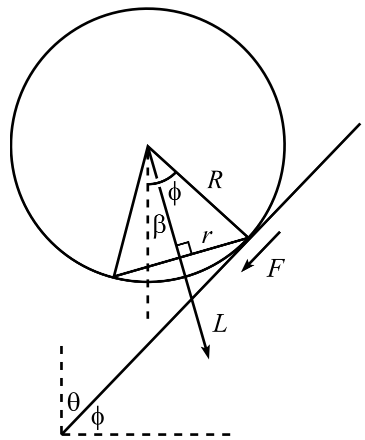
$$\begin{aligned} n \text{ even} \quad \Longrightarrow \quad \begin{pmatrix} v_n \\ R\omega_n \end{pmatrix} &= \frac{n}{7}\begin{pmatrix} 5 & -2 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} V_0 \\ 0 \end{pmatrix}. \\ n \text{ odd} \quad \Longrightarrow \quad \begin{pmatrix} v_n \\ R\omega_n \end{pmatrix} &= \frac{1}{7}\begin{pmatrix} 5n-2 & -2n-2 \\ -5n-5 & 2n-5 \end{pmatrix}\begin{pmatrix} V_0 \\ 0 \end{pmatrix}. \end{aligned} \tag{23}$$

Therefore, the speed along the plane after the nth bounce equals (using $V_0 \equiv 2V\tan\theta$)

$$\begin{aligned} v_n &= \frac{10nV\tan\theta}{7} \qquad (n \text{ even}), \\ v_n &= \frac{(10n-4)V\tan\theta}{7} \qquad (n \text{ odd}). \end{aligned} \tag{24}$$

Remark: Note that after an even number of bounces, eq. (23) gives $v = -R\omega$. This is the "rolling" condition. That is, the angular speed exactly matches up with the translation speed, so v and ω are unaffected by the bounce. (The vector $(1, -1)$ is an eigenvector of $\mathcal{A}$.) At the instant that an even-n bounce occurs, the v and ω are the same as they would be for a ball that simply rolls down the plane. At the instant after an odd-n bounce, the v is smaller than it would be for a rolling ball, but the ω is larger. (And right before an odd-n bounce, the v is larger but the ω is smaller.)

6. It turns out that the ball can move arbitrarily fast around the cone. As we will see, the plane of the contact circle (represented by the chord in the figure below) will need to be tilted downward from the contact point, so that the angular momentum has a rightward horizontal component when it is at the position shown. In what follows, it will be convenient to work with the angle $\phi \equiv 90° - \theta$.



Let's first look at $F = ma$ along the plane. Let Ω be the angular frequency of the ball's motion around the cone. Then the ball's horizontal acceleration is $m\ell\Omega^2$ to the left. So $F = ma$ along the plane gives (where F_f is the friction force)

$$mg\sin\phi + F_f = m\ell\Omega^2\cos\phi. \tag{25}$$

Now let's look at $\boldsymbol{\tau} = d\mathbf{L}/dt$. To get a handle on how fast the ball is spinning, consider what the setup looks like in the rotating frame in which the center of the ball is stationary (so the ball just spins in place as the cone spins around). Since there is no slipping, the contact points on the ball and the cone must have the same speed. That is,

$$\omega r = \Omega\ell \qquad \Longrightarrow \qquad \omega = \frac{\Omega\ell}{r}\,, \tag{26}$$

where ω is the angular speed of the ball in the rotating frame, and r is the radius of the contact circle on the ball.[2] The angular momentum of the ball in the lab frame is $L = I\omega$ (at least for the purposes here[3]), and it points in the direction shown above.

The $\mathbf{L}$ vector precesses around a cone in $\mathbf{L}$-space with the same frequency, Ω, as the ball moves around the cone. Only the horizontal component of $\mathbf{L}$ changes, and it traces out a circle of radius $L_{\text{hor}} = L\sin\beta$, at frequency Ω. Therefore,

$$\left|\frac{d\mathbf{L}}{dt}\right| = L_{\text{hor}}\Omega = (I\omega\sin\beta)\Omega = \frac{I\Omega^2\ell\sin\beta}{r}\,, \tag{27}$$

and the direction of $d\mathbf{L}/dt$ is into the page.

The torque on the ball (relative to its center) is due to the friction force, F_f. Hence, $|\boldsymbol{\tau}| = F_f R$, and its direction is into the page. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives (with

[2] If the center of the ball travels in a circle of radius ℓ, then the ℓ here should actually be replaced with $\ell + R\sin\phi$, which is the radius of the contact circle on the cone. But since we're assuming that $R \ll \ell$, we can ignore the $R\sin\phi$ part.

[3] This $L = I\omega$ result isn't quite correct, because the angular velocity of the ball in the lab frame equals the angular velocity in the rotating frame (which tilts downwards with the magnitude ω we just found) plus the angular velocity of the rotating frame with respect to the lab frame (which points straight up with magnitude Ω). This second part of the angular velocity simply yields an additional vertical component of the angular momentum. But the vertical component of $\mathbf{L}$ doesn't change with time as the ball moves around the cone. It is therefore irrelevant, since we will be concerned only with $d\mathbf{L}/dt$ in what follows.

$I = \eta m R^2$, where $\eta = 2/5$ in this problem)

$$\begin{aligned} F_f R &= \frac{I\Omega^2 \ell \sin\beta}{r} \\ \Longrightarrow \qquad F_f &= \frac{\eta m R \Omega^2 \ell \sin\beta}{r}\,. \end{aligned} \tag{28}$$

Using this F_f in eq. (25) gives

$$mg\sin\phi + \frac{\eta m R\Omega^2 \ell \sin\beta}{r} = m\ell\Omega^2\cos\phi. \tag{29}$$

Solving for Ω gives

$$\Omega^2 = \frac{g\sin\phi}{\ell\left(\cos\phi - \frac{\eta R\sin\beta}{r}\right)}\,. \tag{30}$$

We see that it is possible for the ball to move around the cone infinitely fast if

$$\cos\phi = \frac{\eta\sin\beta}{r/R}\,. \tag{31}$$

If we now define $\gamma \equiv \phi - \beta$ in the above figure, we have $r/R = \sin\gamma$, and $\beta = \phi - \gamma$. So eq. (31) becomes

$$\cos\phi = \frac{\eta\sin(\phi-\gamma)}{\sin\gamma}\,. \tag{32}$$

Using the sum formula for the sine in the numerator, we can rewrite this as

$$\begin{aligned} \tan\gamma = \frac{\eta}{1+\eta}\tan\phi &= \frac{2}{7}\tan\phi \\ &= \frac{2}{7}\cot\theta, \end{aligned} \tag{33}$$

where we have used $\eta = 2/5$. We finally obtain

$$\begin{aligned} \frac{r}{R} = \sin\gamma &= \frac{1}{\sqrt{1+\cot^2\gamma}} \\ &= \frac{1}{\sqrt{1+\frac{49}{4}\tan^2\theta}}\,. \end{aligned} \tag{34}$$

Remarks:

(1) In the limit $\theta \approx 0$ (that is, a very thin cone), we obtain $r/R \approx 1$, which makes sense. The contact circle is essentially a horizontal great circle.

In the limit $\theta \approx 90^\circ$ (that is, a nearly flat plane), we obtain $r/R \approx 0$. The circle of contact points is very small, but the ball can still roll around the cone arbitrarily fast (assuming that there is sufficient friction). This isn't entirely intuitive.

(2) What value of ϕ allows the largest tilt angle of the contact circle (that is, the largest β)? From eq. (31), we see that maximizing β is equivalent to maximizing $(r/R)\cos\phi$,

or equivalently $(r/R)^2 \cos^2\phi$. Using the value of r/R from eq. (34), we see that we want to maximize

$$(r/R)^2 \cos^2\phi = \frac{\cos^2\phi}{1 + \frac{49}{4}\cot^2\phi}\,. \tag{35}$$

Taking the derivative with respect to ϕ and going through a bit of algebra, we find that the maximum is achieved when

$$\tan\phi = \sqrt{\frac{7}{2}} \qquad \Longrightarrow \qquad \phi = 61.9^\circ. \tag{36}$$

You can then show from eq. (31) that

$$\sin\beta_{\text{max}} = \frac{5}{9} \qquad \Longrightarrow \qquad \beta_{\text{max}} = 33.7^\circ. \tag{37}$$

(3) Let's consider three special cases for the contact circle, namely, a horizontal circle, a great circle, and a vertical circle.

(a) *Horizontal circle*: In this case, we have $\beta = 0$, so eq. (30) gives

$$\Omega^2 = \frac{g\tan\phi}{\ell}\,. \tag{38}$$

In this case, $\mathbf{L}$ points vertically, which means that $d\mathbf{L}/dt$ is zero, which means that the torque is zero, which means that the friction force is zero. Therefore, the ball moves around the cone with the same speed as a particle sliding without friction. (You can show that such a particle does indeed have $\Omega^2 = g\tan\phi/\ell$.) The horizontal contact-point circle ($\beta = 0$) is the cutoff case between the sphere moving faster or slower than a frictionless particle.

(b) *Great circle*: In this case, we have $r = R$ and $\beta = -(90^\circ - \phi)$. Hence, $\sin\beta = -\cos\phi$, and eq. (30) gives

$$\Omega^2 = \frac{g\tan\phi}{\ell(1+\eta)}\,. \tag{39}$$

This reduces to the frictionless-particle case when $\eta = 0$, as it should.

(c) *Vertical circle*: In this case, we have $r = R\cos\phi$ and $\beta = -90^\circ$, so eq. (30) gives

$$\Omega^2 = \frac{g\tan\phi}{\ell\left(1 + \frac{\eta}{\cos^2\phi}\right)}\,. \tag{40}$$

Again, this reduces to the frictionless-particle case when $\eta = 0$, as it should. But for $\phi \to 90^\circ$ (thin cone), Ω goes to zero, whereas in the other two cases above, Ω goes to ∞.

2004
Boston Area Undergraduate
Physics Competition

April 24, 2004
Time: 4 hours

Each of the six questions is worth 10 points.

1. (a) (5 points) A sled on which you are riding is given an initial push and slides across frictionless ice. Snow is falling vertically (in the frame of the ice) on the sled. Assume that the sled travels in tracks which constrain it to move in a straight line. Which of the following three strategies causes the sled to move the fastest? The slowest? Explain your reasoning.

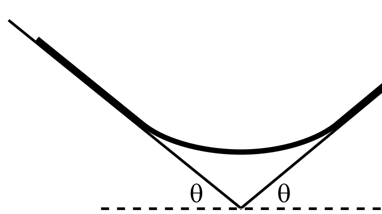
   i. You sweep the snow off the sled so that it leaves the sled in the direction perpendicular to the sled's tracks, as seen by you in the frame of the sled.

   ii. You sweep the snow off the sled so that it leaves the sled in the direction perpendicular to the sled's tracks, as seen by someone in the frame of the ice.

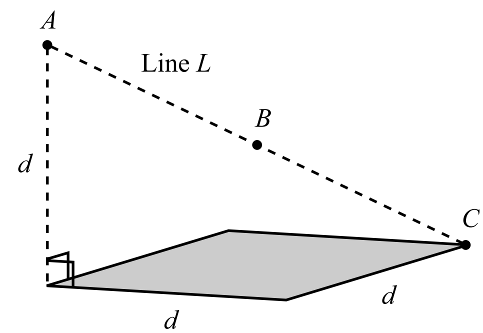
   iii. You do nothing.

   (b) (5 points) You are standing on the edge of a step on some stairs, facing up the stairs. You feel yourself starting to fall backwards, so you start swinging your arms around in vertical circles, like a windmill. This is what people tend to do in such a situation, but does it actually help you not to fall, or does it simply make you look silly? Explain your reasoning.

2. A rope rests on two platforms which are both inclined at an angle θ (which you are free to pick), as shown. The rope has uniform mass density, and its coefficient of friction with the platforms is 1. The system has left-right symmetry. What is the largest possible fraction of the rope that does not touch the platforms? What angle θ allows this maximum value?



3. A flat square plate with side length d serves as a detector for the radiation emitted by a particle. The particle emits the radiation uniformly in all directions. Consider the line, L, joining points A and C, as shown. C is one corner of the square, and A is the point directly above the opposite corner, a distance d above the square.



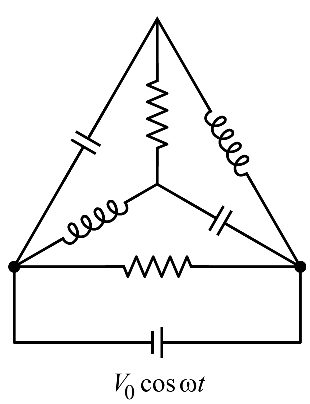
What fraction of the total radiation emitted by the particle is detected by the detector if the particle is placed on the line L:

(a) at point A,

(b) at point B (halfway between A and C),

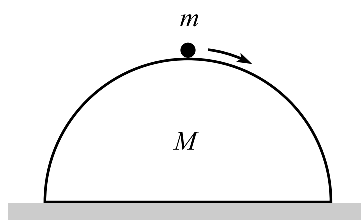
(c) at a point infinitesimally close to point C.

4. The edges of a tetrahedron form an RLC circuit as shown. Two opposite edges are resistors R, two opposite edges are capacitors C, and two opposite edges are inductors L. An alternating voltage with amplitude V_0 is connected to the circuit at the ends of one of the resistors. If the frequency takes the form $\omega = 1/\sqrt{LC}$, and if additionally $R = \sqrt{L/C}$, find the amplitude of the total current through the circuit.

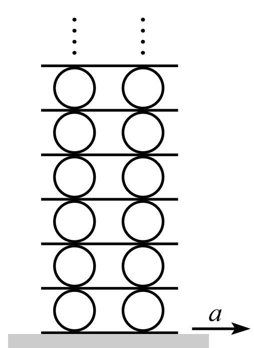


5. A point particle of mass m sits at rest on top of a frictionless hemisphere of mass M, which rests on a frictionless table, as shown. The particle is given a tiny kick and slides down the hemisphere. At what angle θ (measured from the top of the hemisphere) does the particle lose contact with the hemisphere?

   In answering this question for $m \neq M$, it is sufficient for you to produce an equation (please simplify) that θ must satisfy. However, for the special case of $m = M$, your equation can be solved without too much difficulty; find the angle in this case.



6. Consider the infinitely tall system of identical massive cylinders and massless planks shown below. The moment of inertia of the cylinders is $I = MR^2/2$. There are two cylinders at each level, and the number of levels is infinite. The cylinders do not slip with respect to the planks, but the bottom plank is free to slide on a table. If you pull on the bottom plank so that it accelerates horizontally with acceleration a, what is the horizontal acceleration of the bottom row of cylinders?



Solutions

10th Annual
Boston Area Undergraduate
Physics Competition

April 24, 2004

1. (a) From best to worst, the ordering of the strategies is (ii), (iii), (i). We can demonstrate this by using conservation of momentum. There are no external horizontal forces on the sled and the snow, so the total momentum of the sled plus the snow is constant in time.

   Strategy (ii) therefore beats strategy (iii), because the snow in (ii) ends up with no forward momentum, while the snow in (iii) continues to move forward with the sled. The snow in (ii) therefore has less momentum than the snow in (iii), so the sled in (ii) must have more momentum than the sled in (iii).

   Strategy (iii) beats strategy (i) for the following reason. When a snowflake is brushed off the sled in strategy (i), it initially has the same forward speed as the sled as they both sail across the frictionless ice. But when the next snowflake hits the sled, the sled slows down. The brushed-off snowflake therefore now has a larger forward speed than the sled. The sled therefore moves at a speed that is slower than the speed of the center of mass of the sled-plus-snowflake system. But this latter speed is simply the speed of the sled in (iii).

   (b) Swinging your arms does indeed help. Consider the angular momentum of your body relative to your feet. The friction force at your feet provides no torque relative to your feet, so the only external torque is the torque due to gravity (which is what is making you fall over). However, for a small enough period of time, this torque won't angularly accelerate you much, so your angular momentum with respect to your feet is approximately constant.

   Now assume that you start swinging your arms around with the orientation such that your hands are moving forward at the lowest point and backward at the highest point. The right-hand rule then says that your arms have angular momentum which points to your right. But since your angular momentum is approximately constant, there must now be something that has angular momentum pointing to your left. This something is you. You will therefore rotate "forwards" relative to your feet. In other words, you won't fall backwards (assuming that you swing your arms around fast enough).

   Note that it is the *change* in the angular momentum of your arms that is relevant. In other words, the swinging only helps you at the start. Once your arms reach their maximum speed (which in practice happens very quickly), the swinging doesn't help you anymore. But hopefully you've managed to get your center of mass back up above your feet by this time.

2. Let the total mass of the rope be m, and let a fraction f of it hang in the air. Consider the right half of this section. Its weight, $(f/2)mg$, must be balanced by the vertical component, $T\sin\theta$, of the tension at the point where it joins the part of the rope touching the right platform. The tension at that point is therefore $T = (f/2)mg/\sin\theta$.

   Now consider the part of the rope touching the right platform, which has mass $(1-f)m/2$. The normal force from the platform is $N = (1-f)(mg/2)\cos\theta$, so the maximal friction force also equals $(1-f)(mg/2)\cos\theta$, because $\mu = 1$. This fiction force must balance the sum of the gravitational force component along the plane, which is $(1-f)(mg/2)\sin\theta$, plus the tension at the lower end, which we found above. Therefore,

$$\frac{1}{2}(1-f)mg\cos\theta = \frac{1}{2}(1-f)mg\sin\theta + \frac{fmg}{2\sin\theta}\,. \tag{1}$$

   This gives

$$f = \frac{F(\theta)}{1+F(\theta)}\,, \qquad \text{where} \quad F(\theta) \equiv \cos\theta\sin\theta - \sin^2\theta. \tag{2}$$

   This expression for f is a monotonically increasing function of $F(\theta)$, as you can check. The maximal f is therefore obtained when $F(\theta)$ is as large as possible. Using the double-angle formulas, we can rewrite $F(\theta)$ as

$$F(\theta) = \frac{1}{2}(\sin 2\theta + \cos 2\theta - 1). \tag{3}$$

   The derivative of this is $\cos 2\theta - \sin 2\theta$, which equals zero when $\tan 2\theta = 1$. Therefore,

$$\theta_{\text{max}} = 22.5^\circ. \tag{4}$$

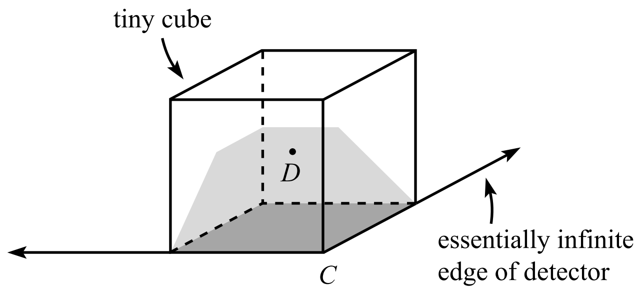
   Eq. (3) then yields $F(\theta_{\text{max}}) = (\sqrt{2}-1)/2$, and so eq. (2) gives

$$f_{\text{max}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = (\sqrt{2}-1)^2 = 3 - 2\sqrt{2} \approx 0.172. \tag{5}$$

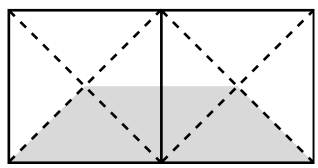
3. Let's consider point B first. This point is the center of a cube of side length d, one of the faces of which is the detector. Since the radiation from the particle is isotropic, 1/6 of it passes through each face of the cube. Therefore, 1/6 of the particle's radiation is detected by the square when the particle is at point B.

   Now consider point A. This point is the center of a cube of side length $2d$. The detector spans one quarter of one of these faces. Combining this fact with the above reasoning tells us that $(1/4)(1/6) = 1/24$ of the particle's radiation is detected by the square when the particle is at point A.

   Lastly, consider a point (call it D) very close to C. This case is a little tricker. Point D is the center of a tiny cube which has as its bottom face a tiny square at the corner of the detector. What areas on this cube correspond to radiation hitting the detector? From the above reasoning, 1/6 of the particle's radiation passes through the bottom face. The other relevant areas on the cube are shown as the lighter shaded regions below.

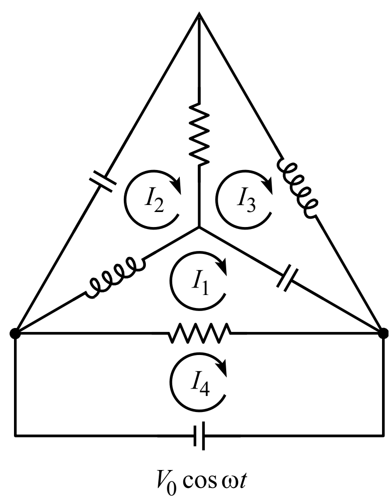


The top horizontal boundary of the lightly shaded region corresponds to the two far edges of the detector, which are essentially infinitely far away. The two diagonal boundaries correspond to the two edges of the detector that emanate from point C.[1] The lightly shaded region covers $3/8$ of the area of the two side faces, as can be seen by flattening out these faces, as shown.



The eight triangles shown in this figure have equal amounts of radiation hitting them, so the shaded $3/8$ of the area corresponds to $3/8$ of the radiation passing through the faces. These two faces represent $1/3$ of the cube, so the total fraction of the particle's radiation that hits the detector is $1/6 + (3/8)(1/3) = 7/24$.

4. We may as well consider the tetrahedron to be a planar circuit, as shown in the diagram below (which looks just like the original 3D diagram). Let the four loop currents be as shown.



[1]These diagonal boundaries are indeed straight lines, which can be seen by noting that each of them is determined by the intersection of two planes, one of which is a face of the tiny cube, and the other of which is the plane determined by an edge of the detector (emanating from point C) and point D.

Since $\omega = 1/\sqrt{LC}$ and $R = \sqrt{L/C}$, the impedances associated with the resistors, inductors, and capacitors take the form,

$$\begin{aligned} Z_R &= R, \\ Z_L &= i\omega L = i\sqrt{\frac{L}{C}} = iR, \\ Z_C &= \frac{-i}{\omega C} = -i\sqrt{\frac{L}{C}} = -iR. \end{aligned} \tag{6}$$

The four loop equations expressing the fact that the voltage drop around a loop is zero are then

$$\begin{aligned} (I_1 - I_4)R + (I_1 - I_2)(iR) + (I_1 - I_3)(-iR) &= 0, \\ I_2(-iR) + (I_2 - I_3)R + (I_2 - I_1)(iR) &= 0, \\ I_3(iR) + (I_3 - I_1)(-iR) + (I_3 - I_2)R &= 0, \\ (I_4 - I_1)R &= V_0. \end{aligned} \tag{7}$$

These simplify to

$$\begin{aligned} (I_1 - I_4) + i(I_3 - I_2) &= 0, \\ (I_2 - I_3) - iI_1 &= 0, \\ (I_3 - I_2) + iI_1 &= 0, \\ (I_4 - I_1) &= V_0/R. \end{aligned} \tag{8}$$

The second and third equations are equivalent, so we in fact have only three equations for our four unknown currents (more on this in the remark below). Multiplying the second equation by i and adding it to the first gives $2I_1 - I_4 = 0 \Longrightarrow I_1 = I_4/2$. Plugging this into the last equation then gives the amplitude of total current through the circuit as

$$I_4 = \frac{2V_0}{R}\,. \tag{9}$$

The effective impedance of the entire circuit is therefore $R/2$, so we see that the upper five lines in the figure effectively act like a resistor of resistance R in parallel with the bottom resistor R.

Remark: The above four equations determine the difference $I_2 - I_3$ to be iV_0/R, but they don't determine I_2 and I_3 individually. These two currents can indeed take on any values, as long as their difference is iV_0/R. Any equal increase in their values simply corresponds to dumping more current on the union of the top two loops (the "2" and "3" loops), which consists of two inductors and two capacitors at resonance (because $\omega = 1/\sqrt{LC}$).

5. Assume that the particle slides off to the right. Let v_x and v_y be its horizontal and vertical velocities, with rightward and downward taken to be positive, respectively. Let V_x be the velocity of the hemisphere, with leftward taken to be positive. Conservation of momentum gives

$$mv_x = MV_x \qquad \Longrightarrow \qquad V_x = \left(\frac{m}{M}\right)v_x. \tag{10}$$

Consider the moment when the particle is located at an angle θ down from the top of the hemisphere. Locally, it is essentially on a plane inclined at angle θ, so the three velocity components are related by

$$\frac{v_y}{v_x + V_x} = \tan\theta \qquad \Longrightarrow \qquad v_y = \tan\theta\left(1 + \frac{m}{M}\right)v_x. \tag{11}$$

To see why this is true, look at things in the frame of the hemisphere. In that frame, the particle moves to the right at speed $v_x + V_x$, and downward at speed v_y. Eq. (11) represents the constraint that the particle remains on the hemisphere, which is inclined at an angle θ at the given location.

Let us now apply conservation of energy. In terms of θ, the particle has fallen a distance $R(1 - \cos\theta)$, so conservation of energy gives

$$\frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}MV_x^2 = mgR(1 - \cos\theta). \tag{12}$$

Using eqs. (10) and (11), we can solve for v_x^2 to obtain

$$v_x^2 = \frac{2gR(1 - \cos\theta)}{(1 + r)\Big(1 + (1 + r)\tan^2\theta\Big)}, \qquad \text{where} \;\; r \equiv \frac{m}{M}. \tag{13}$$

This function of θ starts at zero for $\theta = 0$ and increases as θ increases. It then achieves a maximum value before heading back down to zero at $\theta = \pi/2$. However, v_x *cannot* actually decrease, because there is no force available to pull the particle to the left. So what happens is that v_x initially increases due to the non-zero normal force that exists while contact remains. But then v_x reaches its maximum, which corresponds to the normal force going to zero and the particle losing contact with the hemisphere. The particle then sails through the air with constant v_x. Our goal, then, is to find the angle θ for which the v_x^2 in eq. (13) is maximum. Setting the derivative equal to zero gives

$$\begin{aligned}
&& 0 &= \Big(1 + (1 + r)\tan^2\theta\Big)\sin\theta - (1 - \cos\theta)(1 + r)\frac{2\tan\theta}{\cos^2\theta} \\
\Longrightarrow && 0 &= \Big(1 + (1 + r)\tan^2\theta\Big)\cos^3\theta - 2(1 + r)(1 - \cos\theta) \\
\Longrightarrow && 0 &= \cos^3\theta + (1 + r)(\cos\theta - \cos^3\theta) - 2(1 + r)(1 - \cos\theta) \\
\Longrightarrow && 0 &= r\cos^3\theta - 3(1 + r)\cos\theta + 2(1 + r). \qquad (14)
\end{aligned}$$

This is the desired equation that determines θ. It is a cubic equation, so in general it can't be solved so easily for θ. But in the special case of $r = 1$, we have

$$0 = \cos^3\theta - 6\cos\theta + 4. \tag{15}$$

By inspection, $\cos\theta = 2$ is an (unphysical) solution, so we find

$$(\cos\theta - 2)(\cos^2\theta + 2\cos\theta - 2) = 0. \tag{16}$$

The physical root of the quadratic equation is

$$\cos\theta = \sqrt{3} - 1 \approx 0.732 \qquad \Longrightarrow \qquad \theta \approx 42.9^\circ. \tag{17}$$

Alternate solution: In the reference frame of the hemisphere, the horizontal speed of the particle $v_x + V_y = (1+r)v_x$. The total speed in this frame equals this horizontal speed divided by $\cos\theta$, so

$$v = \frac{(1+r)v_x}{\cos\theta}\,. \tag{18}$$

The particle leaves the hemisphere when the normal force goes to zero. The radial $F = ma$ equation therefore gives

$$mg\cos\theta = \frac{mv^2}{R}\,. \tag{19}$$

You might be concerned that we have neglected the sideways fictitious force in the accelerating frame of the hemisphere. However, the hemisphere is *not* accelerating beginning at the moment when the particle loses contact, because the normal force has gone to zero. Therefore, eq. (19) looks exactly like it does for the familiar problem involving a fixed hemisphere; the difference in the two problems is in the calculation of v.

Using eqs. (13) and (18) in eq. (19) gives

$$mg\cos\theta = \frac{m(1+r)^2}{R\cos^2\theta}\cdot\frac{2gR(1-\cos\theta)}{(1+r)\Big(1+(1+r)\tan^2\theta\Big)}\,. \tag{20}$$

Simplifying this yields

$$\Big(1+(1+r)\tan^2\theta\Big)\cos^3\theta = 2(1+r)(1-\cos\theta), \tag{21}$$

which is the same as the second line in eq. (14). The solution proceeds as above.

Remark: Let's look at a few special cases of the $r \equiv m/M$ value. In the limit $r \to 0$ (in other words, the hemisphere is essentially bolted down), eq. (14) gives

$$\cos\theta = 2/3 \qquad \Longrightarrow \qquad \theta \approx 48.2^\circ, \tag{22}$$

a result which may look familiar to you. In the limit $r \to \infty$, eq. (14) reduces to

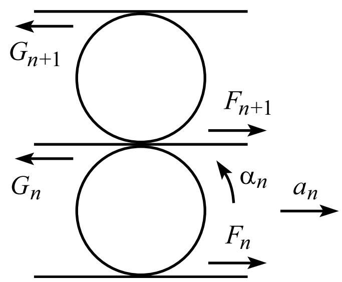
$$0 = \cos^3\theta - 3\cos\theta + 2 \qquad \Longrightarrow \qquad 0 = (\cos\theta - 1)^2(\cos\theta + 2). \tag{23}$$

Therefore, $\theta = 0$. In other words, the hemisphere immediately gets squeezed out very fast to the left.

For other values of r, we can solve eq. (14) either by using the formula for the roots of a cubic equation (very messy), or by simply doing things numerically. A few numerical results are:

r	$\cos\theta$	θ
0	.667	48.2°
1/2	.706	45.1°
1	.732	42.9°
2	.767	39.9°
10	.858	30.9°
100	.947	18.8°
1000	.982	10.8°
∞	1	0°

6. Both cylinders in a given row move in the same manner, so we may simply treat them as one cylinder with mass $m = 2M$. Let the forces that the boards exert on the cylinders be labelled as shown. "F" is the force from the plank below a given cylinder, and "G" is the force from the plank above it.



Note that by Newton's third law, we have $F_{n+1} = G_n$, because the planks are massless.

Our strategy will be to solve for the linear and angular accelerations of each cylinder in terms of the accelerations of the cylinder below it. Since we want to solve for two quantities, we will need to produce two equations relating the accelerations of two successive cylinders. One equation will come from a combination of $F = ma$, $\tau = I\alpha$, and Newton's third law. The other will come from the nonslipping condition.

With the positive directions for a and α defined as in the figure, $F = ma$ on the nth cylinder gives

$$F_n - G_n = ma_n, \tag{24}$$

and $\tau = I\alpha$ on the nth cylinder gives

$$(F_n + G_n)R = \frac{1}{2}mR^2\alpha_n \qquad \Longrightarrow \qquad F_n + G_n = \frac{1}{2}mR\alpha_n \tag{25}$$

Solving the previous two equations for F_n and G_n gives

$$\begin{aligned} F_n &= \frac{1}{2}\left(ma_n + \frac{1}{2}mR\alpha_n\right), \\ G_n &= \frac{1}{2}\left(-ma_n + \frac{1}{2}mR\alpha_n\right). \end{aligned} \tag{26}$$

But we know that $F_{n+1} = G_n$. Therefore,

$$a_{n+1} + \frac{1}{2}R\alpha_{n+1} = -a_n + \frac{1}{2}R\alpha_n. \tag{27}$$

We will now use the fact that the cylinders don't slip with respect to the boards. The acceleration of the board above the nth cylinder is $a_n - R\alpha_n$. But the acceleration of this same board, viewed as the board below the $(n+1)$st cylinder, is $a_{n+1} + R\alpha_{n+1}$. Therefore,

$$a_{n+1} + R\alpha_{n+1} = a_n - R\alpha_n. \tag{28}$$

Eqs. (27) and (28) are a system of two equations in the two unknowns, a_{n+1} and α_{n+1}, in terms of a_n and α_n. Solving for a_{n+1} and α_{n+1} gives

$$\begin{aligned} a_{n+1} &= -3a_n + 2R\alpha_n, \\ R\alpha_{n+1} &= 4a_n - 3R\alpha_n. \end{aligned} \tag{29}$$

We can write this in matrix form as

$$\begin{pmatrix} a_{n+1} \\ R\alpha_{n+1} \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix}. \tag{30}$$

We therefore have

$$\begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix}^{n-1} \begin{pmatrix} a_1 \\ R\alpha_1 \end{pmatrix}. \tag{31}$$

Consider now the eigenvectors and eigenvalues of the above matrix. The eigenvectors are found via[2]

$$\begin{vmatrix} -3-\lambda & 2 \\ 4 & -3-\lambda \end{vmatrix} = 0 \qquad \Longrightarrow \qquad \lambda_\pm = -3 \pm 2\sqrt{2}. \tag{32}$$

The eigenvectors are then

$$\begin{aligned} V_+ &= \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \qquad \text{for } \lambda_+ = -3 + 2\sqrt{2}, \\ V_- &= \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \qquad \text{for } \lambda_- = -3 - 2\sqrt{2}. \end{aligned} \tag{33}$$

Note that $|\lambda_-| > 1$, so $\lambda_-^n \to \infty$ as $n \to \infty$. This means that if the initial $(a_1, R\alpha_1)$ vector has any component in the V_- direction, then the $(a_n, R\alpha_n)$ vectors will head to infinity. This violates conservation of energy. Therefore, the $(a_1, R\alpha_1)$ vector must be proportional to V_+.[3] That is, $R\alpha_1 = \sqrt{2}a_1$. Combining this with the fact that the given acceleration, a, of the bottom board equals $a_1 + R\alpha_1$, we obtain

$$a = a_1 + \sqrt{2}a_1 \qquad \Longrightarrow \qquad a_1 = \frac{a}{\sqrt{2}+1} = (\sqrt{2}-1)a. \tag{34}$$

Remark: Let us consider the general case where the cylinders have a moment of inertia of the form $I = \beta MR^2$. Using the above arguments, you can show that eq. (30) becomes

$$\begin{pmatrix} a_{n+1} \\ R\alpha_{n+1} \end{pmatrix} = \frac{1}{1-\beta} \begin{pmatrix} -(1+\beta) & 2\beta \\ 2 & -(1+\beta) \end{pmatrix} \begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix}. \tag{35}$$

[2] λ_+ happens to be the negative of the $f_{\max}$ result found in Problem 2. An interesting fact, but also a completely random one, I believe.

[3] This then means that the $(a_n, R\alpha_n)$ vectors head to zero as $n \to \infty$, because $|\lambda_+| < 1$. Also, note that the accelerations change sign from one level to the next, because λ_+ is negative.

And you can show that the eigenvectors and eigenvalues are

$$
\begin{aligned}
V_+ &= \begin{pmatrix} \sqrt{\beta} \\ 1 \end{pmatrix}, \qquad \text{for } \lambda_+ = \frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}\,, \\
V_- &= \begin{pmatrix} \sqrt{\beta} \\ -1 \end{pmatrix}, \qquad \text{for } \lambda_- = \frac{\sqrt{\beta}+1}{\sqrt{\beta}-1}\,.
\end{aligned} \tag{36}
$$

As above, we cannot have the exponentially growing solution, so we must have only the V_+ solution. We therefore have $R\alpha_1 = a_1/\sqrt{\beta}$. Combining this with the fact that the given acceleration, a, of the bottom board equals $a_1 + R\alpha_1$, we obtain

$$
a = a_1 + \frac{a_1}{\sqrt{\beta}} \qquad \Longrightarrow \qquad a_1 = \left(\frac{\sqrt{\beta}}{1+\sqrt{\beta}}\right) a. \tag{37}
$$

You can verify that all of these results agree with the $\beta = 1/2$ results obtained above.

Let's now consider a few special cases of the

$$
\lambda_+ = \frac{\sqrt{\beta}-1}{\sqrt{\beta}+1} \tag{38}
$$

eigenvalue, which gives the ratio of the accelerations in any level to the ones in the next level down.

- If $\beta = 0$ (all the mass of a cylinder is located at the center), then we have $\lambda_+ = -1$. In other words, the accelerations have the same magnitudes but different signs from one level to the next. The cylinders simply spin in place while their centers remain fixed. The centers are indeed fixed, because $a_1 = 0$, from eq. (37).
- If $\beta = 1$ (all the mass of a cylinder is located on the rim), then we have $\lambda_+ = 0$. In other words, there is no motion above the first level. The lowest cylinder basically rolls on the bottom side of the (stationary) plank right above it. Its acceleration is $a_1 = a/2$, from eq. (37).
- If $\beta \to \infty$ (the cylinders have long massive extensions that extend far out beyond the rim), then we have $\lambda_+ = 1$. In other words, all the levels have equal accelerations. This fact, combined with the $R\alpha_1 = a_1/\sqrt{\beta} \approx 0$ result, shows that there is no rotational motion at any level, and the whole system simply moves to the right as a rigid object with acceleration $a_1 = a$, from eq. (37).